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Quantum Power Distribution of Relativistic Acceleration Radiation: Classical Electrodynamical Analogies with Perfectly Reflecting Moving Mirrors

Abay Zhakenuly ¹, Maksat Temirkhan ¹, Michael R. R. Good ^{1,*} and Pisin Chen ^{2,3}

¹ Physics Department & Energetic Cosmos Laboratory, Nazarbaev University, Nur-Sultan 010000, Kazakhstan; abay.zhakenuly@nu.edu.kz (A.Z.); maksat.temirkhan@nu.edu.kz (M.T.)

² Leung Center for Cosmology and Particle Astrophysics and Department of Physics and Graduate Institute of Astrophysics, National Taiwan University, Taipei 10617, Taiwan; pisinchen@phys.ntu.edu.tw

³ SLAC National Accelerator Laboratory, Kavli Institute for Particle Astrophysics and Cosmology, Stanford University, Stanford, CA 94305, USA

* Correspondence: michael.good@nu.edu.kz

Abstract: We find the quantum power emitted and distribution in 3 + 1-dimensions of relativistic acceleration radiation using a single perfectly reflecting mirror via Lorentz invariance, demonstrating close analogies to point charge radiation in classical electrodynamics.

Keywords: quantum power distribution; 3 + 1-dimensions; moving mirrors



Citation: Zhakenuly, A.; Temirkhan, M.; Good, M.R.R.; Chen, P. Quantum Power Distribution of Relativistic Acceleration Radiation: Classical Electrodynamical Analogies with Perfectly Reflecting Moving Mirrors. *Symmetry* **2021**, *13*, 653. <https://doi.org/10.3390/sym13040653>

Academic Editor: Stephen A. Fulling, Themis Bowcock

Received: 20 January 2021

Accepted: 26 March 2021

Published: 12 April 2021

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1. Introduction

There are deep connections between point charge radiation in classical electrodynamics and quantum vacuum acceleration radiation from the perfectly reflecting point mirrors of DeWitt–Davies–Fulling [1–3]. For instance in 1982 Ford and Vilenkin [4] demonstrated that the force on a 1 + 1 moving mirror has the same covariant expression as the Lorentz–Abraham–Dirac (LAD) radiation reaction force of an arbitrary moving point charge in 3 + 1 classical electrodynamics. The connection extends to scalar source changes demonstrated, for instance, in 1992 by Higuchi, Matsas and Sudarsky [5] through the discovery that photon emission from a uniformly accelerated classical charge in the Minkowski vacuum corresponds to emission of a zero-energy Rindler photon into an Unruh thermal bath. In 1994, Hai [6] found the quantum energy flux integrated along a large sphere in the asymptotic future as the Larmor formula for the power radiated by a moving scalar charge with respect to an inertial observer. In 2002, Ritus [7–10] found symmetries linking creation of pairs of massless bosons or fermions by an accelerated mirror in 1 + 1 space and the emission of single photons or scalar quanta by electric or scalar charges in 3 + 1 space.

The above studies are just some of the fascinating connections, nowhere near exhaustive, so far discovered. Recent studies ([11–13]) have also strengthened the connection between quantum acceleration radiation and classical radiation of point charges, namely the works by Landulfo, Fulling and Matsas [12], who found that zero-energy Rindler modes are not mathematical artifacts but are critical to understanding the radiation in both the classical and quantum realm, confirming that Larmor radiation emitted by a charge can be seen as a consequence of the Unruh thermal bath. Moreover, Cozzella, Fulling, Landulfo, and Matsas [13] concluded that uniformly accelerated pointlike structureless sources emit only zero-energy Rindler particles using Unruh–deWitt detectors [14]. The analogies hold true with respect to the uniformly accelerated moving mirror, which emits zero energy flux but produces non-zero particle counts as computed from the beta Bogolubov coefficients (e.g., [15]).

Investigations are underway aimed at extending the 1 + 1 dimensional moving mirror model to 3 + 1 dimensions, and understanding the production of scalar particles in the

relativistic regime [16]. These efforts are carried out with the goal of direct detection of relativistic moving mirror radiation [17–19], and they complement the growing accumulation of observations of the dynamical Casimir effect (see references therein [20]). Extension of $1 + 1$ dimensions to $3 + 1$ dimensions, particularly in connection to electrodynamics [21], should be done with caution because, for instance, there is a genuine difference between a two-dimensional world and an effective two-dimensional world arising when a charged particle is placed on a straight line in ambient space. In the former case, the vector potential A behaves in space as r , and hence the charged particle does not radiate at all, whereas in the latter case, A exhibits Coulomb-like behavior, $1/r$, so that usually Larmor radiation results [22].

Experimental verification will be facilitated by knowing the distribution of detected radiation. Recent studies have only just solved for the five classes of uniformly accelerated trajectory distribution [23] in classical electrodynamics (effective Unruh-like temperatures have also been calculated [24]). Unfortunately, mathematically, no settled upon or convincing [25] covariant expression has yet been derived to express the power distribution in a frame-independent formulation [26]. In order to know the distribution of quantum power from a relativistic moving mirror we are also forced to abandon cherished covariant language, while at the same time maintaining the principle of Lorentz invariance.

In this work we compute the quantum power Larmor formula for relativistic moving mirror radiation, the explicit non-covariant quantum power distribution, and apply the results to the simple case of abrupt mirror creation, violent acceleration, and near-instantaneous final constant velocity state of motion for the spectrum of radiation of the mirror. Along the way, we highlight the direct analogies to classical radiation from a moving point charge in electrodynamics.

Our paper is organized as follows: in Section 2, we obtain a total power definition for quantum radiation emitted by a single relativistic moving mirror in $1 + 1$ -dimensions. We highlight that it has identical form to the relativistic generalization of the Larmor formula for power emitted by an accelerated point charge. In Section 3, we derive the angular distribution in time of the quantum radiation of the mirror in $3 + 1$ dimensions using Lorentz invariance. Here we utilize the ansatz that proper acceleration magnitude is a Lorentz scalar independent of direction or dimension. We determine that the quantum power distribution has the same form as in classical electrodynamics. In Section 4, we derive the radiation integral, demonstrating that the approach in electrodynamics equally applies to the mirror. Finally, in Section 5, we specialize our results to an abruptly created, rapidly accelerated, constant velocity moving mirror connecting the frequency-independent spectrum with that of beta decay. We discuss and conclude in Sections 6 and 7, respectively. We use natural units throughout, $\hbar = c = 1$. For conversion from the SI electrodynamics analog, one requires: $q \rightarrow 1$, $\mu_0 \rightarrow 1$, $\epsilon_0 \rightarrow 1$. For the commonly used Gaussian units one converts by $4\pi\epsilon_0 \rightarrow 1$ and $\mu_0 \rightarrow 4\pi$.

2. Relativistic Quantum Larmor Formula

The energy flux, \mathcal{F} , radiated by the mirror moving along a trajectory $p(u)$ in null coordinates where $u = t - x$ and p is the advanced time, $v = t + x$, is derived from the Davies–Fulling quantum stress tensor [2],

$$\mathcal{F}^R(u) = -\frac{1}{24\pi} \{p(u), u\}, \quad (1)$$

where the total energy emitted to the right of the mirror is (e.g., Walker [27]):

$$E^R = \int_{-\infty}^{\infty} \mathcal{F}^R du, \quad (2)$$

defining the Schwarzian derivative by (e.g., Fabbri-Navarro-Salas [28]):

$$\{p(u), u\} = \frac{p'''}{p'} - \frac{3}{2} \left(\frac{p''}{p'} \right)^2. \quad (3)$$

This energy Equation (2) can be expressed in lab coordinates as (see Equation (2.34) of Good, Anderson, and Evans [29]):

$$E^R = \frac{1}{12\pi} \int_{-\infty}^{\infty} \alpha^2 (1 + \dot{x}) dt, \quad (4)$$

where α is the proper acceleration of the mirror. The energy radiated to the left, E^L , is found by the same expression but with a parity flip, $\dot{x} \rightarrow -\dot{x}$, so that the total radiated energy is:

$$E = E^R + E^L = \frac{1}{6\pi} \int_{-\infty}^{\infty} \alpha^2 dt. \quad (5)$$

This allows us to identify and define a relativistic quantum power for the moving mirror, $P \equiv dE/dt$,

$$E = \int_{-\infty}^{\infty} \frac{dE}{dt} dt \equiv \int_{-\infty}^{\infty} P dt, \quad (6)$$

which gives, from Equation (5), a familiar acceleration scaling:

$$P_{\text{mirror}} = \frac{\alpha^2}{6\pi}. \quad (7)$$

The quantum vacuum scalar radiation power, Equation (7), (in SI units $P_{\text{mirror}} = \hbar\alpha^2/6\pi c^2$) emitted by the mirror enjoys the same scaling as the relativistic Larmor formula,

$$P_{\text{electron}} = \frac{2}{3} \frac{q^2 \alpha^2}{4\pi\epsilon_0 c^3} = \frac{q^2 \alpha^2}{6\pi\epsilon_0 c^3} \rightarrow \frac{q^2 \alpha^2}{6\pi}, \quad (8)$$

in classical electrodynamics, where we start in SI units and “ \rightarrow ” implies conversion to natural units where $\epsilon_0 = c = 1$, and α is the magnitude of the proper acceleration of the moving point charge (e.g., electron).

3. Angular Distribution in Time

The relativistic quantum generalization of Larmor’s power formula, Equation (7), is a Lorentz invariant scalar,

$$P = P^* = \frac{\alpha^2}{6\pi'}, \quad (9)$$

where P^* is the instant rest frame power and P is the lab frame power. Here we make a critical assumption about the 1 + 1-dimensional result (Equation (9)): the universality of the Lorentz invariant scalar in any frame suggests it is independent of dimension. It is just a number after all, with no associated direction (see Appendix A for elucidation). We take this as an ansatz and find it possible to proceed. In this case, a simple definition for the angular distribution of the power Equation (9) of a moving mirror in 3 + 1 dimensions can be written as (we imagine the boundary condition is still defined as a perfectly reflecting point, not a two-dimensional surface, but see the end of this section for more discussion):

$$P = \int \frac{dP^*}{d\Omega^*} d\Omega^* = \int \frac{dP^*}{d\Omega^*} \sin \Theta^* d\Theta^* d\phi^*. \quad (10)$$

Here the $*$ designates the instantaneous rest frame, and the angle Θ^* is between the acceleration 3-vector \mathbf{a}^* and the direction to the observer in the $*$ frame. What we have done is assumed that the power formula derived in 1 + 1 dimensions (Equation (9)) holds true in 3 + 1 dimensions (this was found 20 years ago by Bekenstein for power emitted by

black holes in the context of information transmission [30]). Likewise we will assume the power is distributed in 3 + 1 dimensions by using the 3 + 1 dimensional Lorentz covariant definition of proper acceleration. Therefore:

$$\frac{dP^*}{d\Omega^*} = \frac{|\mathbf{a}^* \times \hat{\mathbf{n}}^*|^2}{16\pi^2} = \frac{\alpha^2 \sin^2 \Theta^*}{16\pi^2}, \quad (11)$$

where the instantaneous rest frame unit vector, $\hat{\mathbf{n}}^*$, is:

$$\hat{\mathbf{n}}^* = \sin \theta^* \cos \phi^* \hat{x}^* + \sin \theta^* \sin \phi^* \hat{y}^* + \cos \theta^* \hat{z}^*, \quad (12)$$

and \mathbf{a}^* is the proper acceleration 3-vector in the instant rest frame, where $\alpha^2 \equiv -a_\mu a^\mu$. The acceleration 4-vector in the instant rest frame has components $(0, \mathbf{a}^*)$, i.e., α is the invariant Lorentz scalar, the acceleration felt by the moving mirror itself—its “property” [31]. The numerical $16\pi^2$ factor in Equation (11) originates from:

$$\int_0^{2\pi} \int_0^\pi \frac{\sin^2 \Theta^*}{16\pi^2} \sin \Theta^* d\Theta^* d\phi^* = \frac{1}{16\pi^2} \frac{8\pi}{3} = \frac{1}{6\pi}. \quad (13)$$

See Figure 1 for an illustration of the usual angle convention we have adopted. We now wish to find the power distribution starting from Equation (11). We will find the dimensional allocation without appeal to fields or potentials, using only the principle of Lorentz invariance.

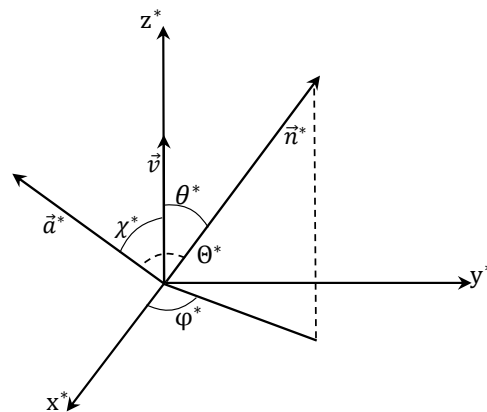


Figure 1. This figure shows the angles we adopt between the relevant vectors, where χ^* is the angle between acceleration and velocity in the x^*z^* plane, θ^* is the angle between velocity and the normal vector, Θ^* is the angle between acceleration and the normal vector, and ϕ^* is the azimuthal angle.

First, let us transform the solid angle to the lab frame. Under Lorentz transform the solid angle is (e.g., [25]):

$$d\Omega^* = \frac{d\Omega}{\gamma^2(1 - \beta \cos \theta)^2}, \quad (14)$$

and the energy density is $dU^* = \gamma(1 - \beta \cos \theta)dU$, while the square of the acceleration scalar is written,

$$\alpha^2 = \gamma^6(a^2 - (\boldsymbol{\beta} \times \mathbf{a})^2). \quad (15)$$

Hence, we write, using $dt = \gamma dt^*$, the distribution from the source, Equation (11), in the lab frame coordinates

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1 - \beta \cos \theta)^3} \frac{dP^*}{d\Omega^*}, \quad (16)$$

obtaining:

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2} \frac{\gamma^2[a^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2]}{(1 - \beta \cos \theta)^3} \sin^2 \Theta^*. \quad (17)$$

Since the distribution is needed in the lab frame, the rest frame angle Θ^* should be described in terms of lab frame angles θ , ϕ , and χ (see Appendix C). To put Equation (17) in a more explicitly useful (and recognizable) form without reference to the angle Θ^* , we would like to show that:

$$g^2 \gamma^2 [a^2 - (\mathbf{v} \times \mathbf{a})^2] \sin^2 \Theta^* = [\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a})]^2, \tag{18}$$

or more concisely, that:

$$|\mathbf{a}^* \times \hat{\mathbf{n}}^*|^2 = \frac{\gamma^4}{g^2} [\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a})]^2. \tag{19}$$

To do this, we express the instantaneous rest frame 3-vector acceleration, \mathbf{a}^* , in terms of the lab frame 3-vector acceleration, \mathbf{a} , via the appropriate usual Lorentz acceleration transformation (e.g., [32]):

$$\mathbf{a}^* = \frac{\mathbf{a}}{\gamma^2(1-v^2)^2} - \frac{(\mathbf{a} \cdot \mathbf{v})\mathbf{v}(\gamma-1)}{v^2\gamma^2(1-v^2)^3} + \frac{(\mathbf{a} \cdot \mathbf{v})\mathbf{v}}{\gamma^2(1-v^2)^3}. \tag{20}$$

We can transform instant rest angles (θ^*, ϕ^*) to lab angles (θ, ϕ) by:

$$\sin^2 \theta^* = \frac{\sin^2 \theta}{\gamma^2 g^2}, \quad \cos^2 \theta^* = \frac{(\cos \theta - \beta)^2}{g^2}, \tag{21}$$

where g is the Doppler factor, $g \equiv 1 - \beta \cos \theta$ and $\phi^* = \phi$. This gives:

$$|\mathbf{a}^* \times \hat{\mathbf{n}}^*|^2 = |\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a}) - \mathbf{a} - \mathbf{v}(\hat{\mathbf{n}} \cdot \mathbf{a}) + \mathbf{a}(\hat{\mathbf{n}} \cdot \mathbf{v})|^2 \frac{\gamma^4}{g^2}. \tag{22}$$

From $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, the right-hand side of Equation (22) is equal to the right-hand side of Equation (19). Therefore, our power distribution in the lab frame, Equation (17), is (see Appendix D for a demonstration of well-known limits):

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2} \frac{[\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})]^2}{(1 - \beta \cos \theta)^5}. \tag{23}$$

This form for the quantum power distribution of scalar radiation from a relativistic moving mirror is that of the widely used textbook result for the classical power distribution of an arbitrarily moving electric point charge (e.g., [33–35]). Note that this derivation does not rely upon fields or potentials and depends solely on the validity of Lorentz invariance of proper acceleration and its vector decomposition in 3 + 1 dimensions. Although the derivation method of Equation (23) was for quantum radiation from a moving mirror, it is also valid for classical radiation from a moving electron.

The exact equation of motion of the system in 3 + 1 dimensions has not been defined. The derivation does not rely upon the fields directly, but the system obeys the homogeneous Klein–Gordon equation everywhere except at the mirror and some boundary conditions at the mirror. The main result of this paper is to demonstrate that the constraints of Lorentz invariance and dependence only on acceleration completely determine the radiation distribution, given the single integrated power quantity Equation (9). Barring the details of the (1 + 1)-dimensional derivation of Equation (9), the (3 + 1)-dimensional result is independent of the precise boundary condition at the mirror (or whatever is located at the singular point).

4. Angular Distribution in Frequency

In this section we confirm a procedure to obtain the widely-used classical radiation integral of electrodynamics but for the case of quantum scalar radiation from a moving mirror. This is needed to find the angular distribution in frequency. We demonstrate that

the derivation and assumptions applied in the context of classical electromagnetic fields is applicable to the quantum scalar field.

Converting to the frequency domain requires considering the power distribution in the time domain as an energy density distribution in both time and angle, and defining angular distribution in frequency,

$$\frac{dP(t)}{d\Omega} = \frac{d^2U}{dt d\Omega}, \quad \frac{dI(\omega)}{d\Omega} = \frac{d^2U}{d\omega d\Omega}. \quad (24)$$

where we write:

$$\frac{dU}{d\Omega} = \int_{-\infty}^{+\infty} dt \frac{dP(t)}{d\Omega} = \int_0^{\infty} d\omega \frac{dI(\omega)}{d\Omega}. \quad (25)$$

Parseval's theorem can be used to deduce that:

$$\frac{dU}{d\Omega} = \int_{-\infty}^{+\infty} dt \frac{dP(t)}{d\Omega} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{dP(\omega)}{d\Omega}. \quad (26)$$

We apply a reality condition to the integrand such that $P(t) = P^*(t)$ and $P(\omega) = P^*(-\omega)$, which gives an even integral,

$$\frac{dU}{d\Omega} = \frac{1}{\pi} \int_0^{+\infty} d\omega \frac{dP(\omega)}{d\Omega}. \quad (27)$$

The angular distribution in frequency can be found by a derivative,

$$\frac{d}{d\omega} \frac{dU}{d\Omega} = \frac{d}{d\omega} \frac{1}{\pi} \int d\omega \frac{dP(\omega)}{d\Omega}, \quad (28)$$

which amounts to, using the notation of Equation (24):

$$\frac{dI(\omega)}{d\Omega} = \frac{d^2U}{d\omega d\Omega} = \frac{1}{\pi} \frac{dP(\omega)}{d\Omega}. \quad (29)$$

We now have picked up a π and a Fourier transform. We plug in Equation (23), which is the time-domain Equation (24), into:

$$\frac{dI(\omega)}{d\Omega} = \int_{-\infty}^{+\infty} dt \frac{dP(t)}{d\Omega} e^{i\phi}, \quad (30)$$

where $\phi = \omega(t_r - \hat{n} \cdot \mathbf{r}_0(t_r))$, using the radiation zone approximation. This assumes the observation point is very far from the regions of space where the acceleration is non-zero: $\hat{n}_r \approx \hat{r}$ is constant. The mirror always moves on some arbitrary trajectory $\mathbf{r}_0(t)$ with velocity $\mathbf{v}(t) = \dot{\mathbf{r}}_0(t)$. Then, using Equation (29), as well as expressing the Fourier transform over retarded time, $dt = g dt_r$, gives the widely-used analog formula (see, e.g., Jackson [33] or Zangwill [34]):

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{16\pi^3} \left| \int_{-\infty}^{\infty} dt_r \frac{[\hat{n} \times ((\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})] e^{i\phi}}{(1 - \beta \cos \theta)^2} \right|^2. \quad (31)$$

An important virtue of the form of Equation (31) is that the integrand is zero when the mirror acceleration is zero, which will always be the case for collision scattering-type and open-orbit situations where the mirror is subjected to a force for a finite amount of time. In the next section we will look specifically at such a situation in the case of abrupt creation of the mirror itself.

5. Specialized Case: Abrupt Mirror Creation

The sudden creation of a fast moving mirror can be viewed, for our purposes, as the violent acceleration of the mirror initially at rest to some final velocity within a very short time interval, or, alternatively, as the sudden switching on of the boundary describing the moving mirror in the same short time interval.

Consider the initial situations, where the the mirror is either static or non-existent: there is no radiation. Now consider the more subtle final situation, where a Lorentz boost to the frame of the moving mirror renders the final constant velocity state of motion as trivially non-radiating as that of a static or non-existent mirror. Thus, it is the intermediate situation, where the radiation is sourced from the non-trivial physics during the short time period, in which there is violent acceleration or sudden creation.

Our goal is to calculate the spectrum of this perfectly reflecting boundary. While our calculation can be interpreted both as an accelerated boundary with violent acceleration to constant velocity, and the sudden creation of the mirror, these systems are not the same! As pointed out in the previous sections, the specification of the particular equation of motion and boundary condition in $3 + 1$ dimensions is not given. Examples can be found in previous work, such as that by Brown and Louko, who described the case of smooth creation of a two-sided Dirichlet mirror in $(1 + 1)$ -dimensional flat space-time that generates a flux of real quanta [19]. Also relevant is the creation of boundaries in $3 + 1$ dimensions and the resulting particle production; that induced by the creation that changes the gravitational field itself as worked out for cosmic strings by Parker [36] (unlike a point Dirichlet mirror, it is known that cosmic strings have effects on the quantum fields around them in $3 + 1$ dimensions). For our particular spectral result, the systems cannot be distinguished from each other. That is to say, while sudden creation and violent acceleration are not the same physical processes, the nature of our approximation gives a regime of validity where the result can be interpreted as either sudden creation or violent acceleration. The spectrum will critically depend on a short time period approximation of Equation (31), where all that matters is before $t < 0$ there is no radiation contribution to the integral and after $t > 0$ the mirror is moving at constant velocity relative to a faraway observer. It is this limiting short-period form that can be straightforwardly integrated.

To compute the spectrum, we first start with the use of a perfect differential identity (see Appendix B for a proof):

$$\frac{\hat{n} \times ((\hat{n} - \beta) \times \dot{\beta})}{(1 - \beta \cos \theta)^2} = \frac{d}{dt_r} \left[\frac{\hat{n} \times (\hat{n} \times \beta)}{1 - \beta \cos \theta} \right], \quad (32)$$

where the derivatives are evaluated at retarded time. We apply this identity to perform an integration by parts on Equation (31),

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{16\pi^3} \left| \frac{\hat{n} \times (\hat{n} \times \beta)}{1 - \beta \cos \theta} e^{i\phi} \right|_0^{+\infty} - i\omega \int_0^{\infty} dt_r \hat{n} \times (\hat{n} \times \beta) e^{i\phi} \Big|^2, \quad (33)$$

where the boundary terms vanish. Again, we have defined $\phi = \omega t_r - \mathbf{k} \cdot \mathbf{r}_0(t_r)$, where $k = \omega$. Using $\hat{n} \approx \hat{r}$ as a constant vector means it comes outside the integral, and since $|\hat{r} \times (\hat{r} \times \beta)|^2 = |\hat{r} \times \beta|^2$ for any vector β , we obtain the angular spectrum of radiated energy as:

$$\frac{dI(\omega)}{d\Omega} = \frac{\omega^2}{16\pi^3} \left| \hat{r} \times \int_0^{\infty} dt \beta(t) \exp[-i(\mathbf{k} \cdot \mathbf{r}_0(t) - \omega t)] \right|^2. \quad (34)$$

The integral is zero from $-\infty$ to 0 but non-zero from 0 to $+\infty$. The non-zero contribution comes because $\beta(t) = \beta$ for $t > 0$ with trajectory function $r_0(t) = \beta t$. Using $\mathbf{k} = \omega \hat{\mathbf{r}}$,

$$\frac{dI(\omega)}{d\Omega} = \frac{\omega^2}{16\pi^3} |\hat{\mathbf{r}} \times \boldsymbol{\beta}|^2 \left| \int_0^\infty dt \exp[-i\omega(\hat{\mathbf{r}} \cdot \boldsymbol{\beta} - 1)t] \right|^2. \quad (35)$$

Notice the pre-factor ω^2 frequency dependence, which will ultimately cancel out after appropriate integration. The integral diverges at late times (upper limit), so we use a convergence regulator $e^{-\epsilon t}$ and set $\epsilon \rightarrow 0$ after integration. Using $\hat{\mathbf{r}} \cdot \boldsymbol{\beta} = \beta \cos \theta$, where θ is the angle between $\boldsymbol{\beta}$ and the observation point, the integral is:

$$\int_0^\infty dt \exp[-i\omega(\beta \cos \theta - 1)t - \epsilon t] = \frac{i}{i\epsilon + \omega(1 - \beta \cos \theta)}, \quad (36)$$

where we can now set $\epsilon \rightarrow 0$, and write the square of the integral as:

$$\left| \int_0^\infty dt \exp[-i\omega(\hat{\mathbf{r}} \cdot \boldsymbol{\beta} - 1)t] \right|^2 = \frac{1}{\omega^2(1 - \beta \cos \theta)^2}, \quad (37)$$

demonstrating that frequency dependence cancels out exactly from Equation (35). Using $\hat{\mathbf{r}} \times \boldsymbol{\beta} = \beta \sin \theta$, the result is:

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{16\pi^3} \frac{\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)^2}. \quad (38)$$

This is angular distribution of energy radiated per unit frequency in the frequency domain for the abrupt creation or violent acceleration of a moving mirror in $3 + 1$ dimensions. We can see that in this application, the angular distribution of energy radiated per unit frequency (and the total energy radiated per unit frequency in the next subsection) are independent of frequency.

Total Energy & Particles

The total energy radiated per unit frequency is found by the integral

$$I(\omega) = \frac{1}{16\pi^3} \int d\Omega \frac{\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)^2} \quad (39)$$

Using $d\Omega = \sin \theta d\theta d\phi$, the numerator scales by $\sin^3 \theta$ and integrating θ from $(0, \pi)$ and ϕ from $(0, 2\pi)$, we obtain: a simple expression in terms of the final rapidity, $\eta = \tanh^{-1} \beta$, and final velocity, β , of the mirror,

$$I(\omega) = \frac{1}{2\pi^2} \left(\frac{\eta}{\beta} - 1 \right). \quad (40)$$

For non-relativistic speeds $\beta \ll 1$, the radiated intensity is negligible and scales as $I(\omega) = \beta^2/(6\pi^2)$. For ultra-relativistic speeds, $I(\omega) = \eta/(2\pi^2)$. See Figure 2 for a plot of Equation (40).

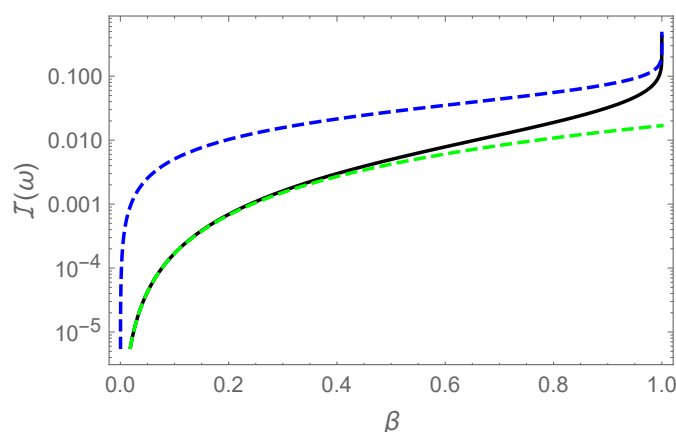


Figure 2. A plot of the spectrum, $I(\omega)$, for the violent acceleration of a moving mirror to constant velocity, Equation (40), as a function of the final speed, β . Note that the radiated intensity is negligible at non-relativistic speeds (green) scaling as β^2 but at ultra-relativistic speeds (blue) the spectrum scales as the rapidity η . Equation (40) corresponds to the spectrum of an electron's radiation during beta decay.

Ultimately, the spectrum does not depend on the frequency ω because the mirror is made to be in instantaneous motion at $t = 0$ with velocity v . A more physical picture will have the velocity approached in some very short time interval Δt . In this case the spectrum will die off and be negligible when $\omega \gg 1/\Delta t$. The number of scalars per unit energy range is given by:

$$N(\omega) = \frac{1}{2\pi^2\omega} \left(\frac{\eta}{\beta} - 1 \right). \quad (41)$$

Their total energy radiated has a maximum frequency dependence:

$$E_{\text{rad}} = \int_0^{\omega_{\text{max}}} \omega N(\omega) d\omega = \frac{1}{2\pi^2} \left(\frac{\eta}{\beta} - 1 \right) \omega_{\text{max}}. \quad (42)$$

We reiterate that the mirror is assumed to be created with constant velocity such that the acceleration period occurs within a very short time interval. The spectrum function is negligibly small when the frequency is much larger than the inverse time interval of acceleration. The intensity distribution, Equation (40), has the same form as a typical inner bremsstrahlung spectrum [33] in classical electrodynamics. In the case of beta decay [33,34], e.g., the electron plays the role of the mirror.

6. Discussion

First, we should comment on the regime of validity of $P = \alpha^2/(6\pi)$ and the assumptions that were required to obtain it. Equation (4) has undergone an integration by parts (for details see Equation (2.34) of [29]), which is only valid if globally the acceleration of the mirror is asymptotically zero in both the past and future. Moreover, the boundary terms only disappear if the mirror is sub-light speed asymptotically, which is an even stronger constraint.

This seemingly inconsequential subtlety almost certainly has much to do with the longstanding debate [37,38] over whether a uniformly accelerated point charge radiates (consider the clear distinctions needed between the electromagnetic power received by a set of far-off observers and the instantaneous mechanical power loss of the charge in [39]); in this case we comment that the power formula does not apply to global uniform acceleration, as that would violate the previously mentioned assumptions giving non-vanishing boundary terms. That is, a globally uniformly accelerated mirror does not radiate [15] energy but does radiate particles (as is well known [40–42]). This is also in

line with the fact that uniformly accelerated point-like structure-less sources emit only zero-energy Rindler particles [13].

Second, we comment on the assumption that the Lorentz power scalar holds in higher dimensions. Conformal invariance breaks down in higher dimensions [18] but our derivation, underscoring the Larmor formula as a Lorentz scalar independent of direction, does not ostensibly require conformal invariance. The moving mirror in $1 + 1$ dimensions permits exact solutions to the field equation because of conformal invariance but our result does not appeal to an exact explicit solution to the wave equation of motion. It only appeals to the dynamics of the mirror as computed through the general renormalized quantum stress tensor as given by the Schwarzian derivative. In general contexts, the quantum stress tensor, as opposed to particle production, is much easier to obtain. This is true in higher dimensions where conformal invariance no longer holds (see [18] and references therein, e.g., Refs [15–19]).

Third, we comment on the general applicability of the power distribution of Equation (23). This formula applies to very general trajectories (albeit globally asymptotically sub-light speed and time-like; although this does not stop one from being able to locally compute the power distributions of the five classes of uniform acceleration, which for an electric charge will be the same form as for uniformly accelerated moving mirrors [23]. It is unclear whether the effective temperatures will also carry-over [24]). For instance, a mirror moving on a circular arc near the speed of light will emit a synchrotron-type of radiation in the form of a narrow and intense beam directed tangent to the arc, implying that a fixed observer will see a brief flash or pulse of radiation every time the mirror moves directly toward them.

This power distribution, Equation (23), in the context of vacuum acceleration radiation, warrants study because of its importance in connection to synchrotron radiation. Disentanglement in contexts where it is relevant will be essential in confirming the source's origin. While astrophysical synchrotron radiation is a powerful indicator of the presence of magnetic fields and particle acceleration mechanisms near pulsars and black holes, the possibility that similar radiation could point to quantum amplification of vacuum fluctuations due to an accelerated boundary condition is of interest to a wide range of physicists concerned with relativistic quantum fields and information.

7. Conclusions

We have investigated the acceleration radiation emitted by a single relativistic perfect point mirror in $3 + 1$ dimensions and its analogies with a moving point charge in electrodynamics. Namely,

- We found the quantum power formula for moving mirrors and identified it with Larmor's form.
- The quantum Larmor formula is not applicable for eternal uniform acceleration.
- We generalized the $1 + 1$ -dimensional moving mirror model to $3 + 1$ dimensions in the context of distributed power. This was done with the ansatz that the scalar power in $1 + 1$ dimensions (which is proportional to the proper acceleration magnitude squared) is also the scalar power in $3 + 1$ dimensions, i.e., the scalar will be invariant under Lorentz transformations in $3 + 1$ dimensions and obeys Larmor's form. The only covariant choice of definition for scalar magnitude of proper acceleration in $3 + 1$ dimensions is that constructed by the usual Lorentz 4-vector acceleration.
- Consequently, we derived the power distribution and found its form is in analog to the well-known power distribution in electrodynamics (e.g., the apportioning responsible for synchrotron radiation.). This was done by Lorentz invariant vector decomposition of relativistic acceleration. The allotment is a result of the motion of the mirror only and the derivation did not rely on the use of fields or potentials, i.e., there are no requisite Lienard–Wiechert potentials or electric and magnetic field counterparts for the quantum scalar radiation.

- We derived the spectrum of a moving mirror abruptly created and violently accelerated to a constant velocity. In analog to beta decay in classical electrodynamics, we found the appearance of the moving mirror plays the role of the moving electron.

Knowing the radiated distribution of power for the moving mirror is a necessary first step toward precise orientation for accurate detection. The work here determines that the distribution for quantum scalar radiation from a moving mirror is the same form as that of accelerated electron radiation in classical electrodynamics. The close analogy suggests direction for future work, including the application and interpretation of moving mirror trajectories in analog to the exactly known spectra associated with the moving point charge, including, for example, and not limited to, the well-known bending magnet trajectory, the undulator trajectory, and the collinear acceleration burst trajectory.

Author Contributions: All authors contributed to this work. All authors have read and agreed to the published version of the manuscript.

Funding: Funding from the state-targeted program “Center of Excellence for Fundamental and Applied Physics” (BR05236454) by the Ministry of Education and Science of the Republic of Kazakhstan is acknowledged, as well as the FY2021-SGP-1-STMM Faculty Development Competitive Research Grant No. 021220FD3951 at Nazarbayev University.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Comment on Invariant Scaling

A Lorentz scalar is a number that is invariant under Lorentz transformations. The most well-known examples include the speed of light c , the space-time distance between two fixed events Δs^2 , rest mass m_0 , proper time τ , and $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ in electrodynamics.

Numbers that are not invariant under Lorentz transformations, which we could call “non-Lorentz” scalars, may have no associated direction but nevertheless change under a Lorentz transformation. Examples include $\mathbf{E} \cdot \mathbf{E}$ or electric charge density, which are invariant with respect to spatial rotations but not with respect to boosts. Other examples are components of vectors and tensors that in general are altered under Lorentz transformations.

A Lorentz scalar is invariant in a given dimensionality, but it is not a priori guaranteed that the same physics in a different dimension would render the same scaling. Consider the Stefan–Boltzmann law that scales as $P \sim T^2$ in $(1 + 1)\text{D}$ and $P \sim T^4$ in $(3 + 1)\text{D}$. Despite no associated direction, the physics changes dramatically in different dimensions (see below for a caveat). This may be an example of a number which is not invariant under Lorentz transformation—a “non-Lorentz” scalar (see, e.g., the relativistic Stefan–Boltzmann law [43]). There is no consensus on whether temperature is a Lorentz scalar. (Einstein [44] and Planck [45] derived a moving body to be cooler, $T' = T/\gamma$, whereas Ott [46] derived a moving body to be warmer, $T' = \gamma T$. Landsberg [47] derived $T' = T$).

Our ansatz that the quantum power remains invariant under a change of dimensions, used to derive Equation (10), is motivated by the invariance of the proper acceleration as a Lorentz scalar. This assumption is akin to conjecturing that the speed of light remains the same in both $(1 + 1)\text{D}$ and $(3 + 1)\text{D}$. While our conjecture is in no way guaranteed, it is a good starting point for the form of the $(3 + 1)\text{D}$ quantum power.

The caveat to the Stefan–Boltzmann scaling is seen by considering a $(1 + 1)\text{D}$ thermal system (like the thermal moving mirror [48] with energy flux $\mathcal{F} = \pi T^2/12$), where the power is thus:

$$P \sim T^2. \quad (\text{A1})$$

However, for a black body surface in $(3 + 1)\text{D}$, using the usual Stefan–Boltzmann law, we have:

$$P \sim AT^4, \quad (\text{A2})$$

which, as we have just mentioned, seemingly deviates from Equation (A1). However, for a Schwarzschild black hole of temperature $T \sim M^{-1}$, we see the area is $A \sim M^2$ or $A \sim T^{-2}$.

Substituting this into Equation (A2) we retrieve Equation (A1). Therefore, a (3 + 1)D black hole acts as a (1 + 1)D moving mirror with the same power dependence on temperature (accelerated boundary correspondences exist for more than just the Schwarzschild black hole [49]. See the Reissner–Nordström [50], Taub-NUT [51] and Kerr [52] black holes. De Sitter and anti-de Sitter cosmologies [53] are also modeled by moving mirrors as well as extremal black holes [54–56]). This interesting caveat to the change in dimensional scaling provides an example where the power does remain invariant for a moving mirror.

Appendix B. Derivation of Perfect Differential Identity

Here we derive the identity Equation (32) in the text. We start with the right-hand side of Equation (32), where we have dropped the subscript of retarded time,

$$\frac{d}{dt} \left[\frac{\hat{n} \times (\hat{n} \times \beta)}{1 - \beta \hat{n}} \right] = \frac{(\hat{n} \times (\hat{n} \times \beta))'(1 - \beta \hat{n}) - (1 - \beta \hat{n})'(\hat{n} \times (\hat{n} \times \beta))}{(1 - \beta \hat{n})^2}, \tag{A3}$$

using vector multiplication as the dot product (i.e., $\beta \hat{n} \equiv \beta \cdot \hat{n}$) unless otherwise indicated. Calculating the numerator using the bac-cab rule and then differentiating,

$$[(\hat{n} \times (\hat{n} \times \beta))]' = [(\hat{n}\beta)\hat{n} - (\hat{n}\hat{n})\beta]' = (\hat{n}\dot{\beta})\hat{n} - \dot{\beta}, \tag{A4}$$

and differentiating the Doppler factor g ,

$$[1 - \beta \hat{n}]' = -\dot{\beta} \hat{n}, \tag{A5}$$

hence the numerator expanded out and simplified gives:

$$\begin{aligned} & ((\hat{n}\dot{\beta})\hat{n} - \dot{\beta})(1 - \beta \hat{n}) - (-\dot{\beta} \hat{n})((\hat{n}\beta)\hat{n} - (\hat{n}\hat{n})\beta) \\ &= (\hat{n}\dot{\beta})\hat{n} - (\beta \hat{n})(\dot{\beta} \hat{n}) - \dot{\beta} + \dot{\beta}(\beta \hat{n}) + (\beta \hat{n})(\dot{\beta} \hat{n}) - (\dot{\beta} \hat{n})\beta \\ &= -\dot{\beta} - (\dot{\beta} \hat{n})\beta + (\hat{n}\dot{\beta})\hat{n} + \dot{\beta}(\beta \hat{n}), \end{aligned} \tag{A6}$$

such that our time derivative gives a form inversely proportional to the Doppler factor squared:

$$\frac{d}{dt} \left[\frac{\hat{n} \times (\hat{n} \times \beta)}{1 - \beta \hat{n}} \right] = \frac{-\dot{\beta} - (\dot{\beta} \hat{n})\beta + (\hat{n}\dot{\beta})\hat{n} + \dot{\beta}(\beta \hat{n})}{(1 - \beta \hat{n})^2}. \tag{A7}$$

The denominators of both sides of Equation (32) are now equal. Consider the numerator only of the left-hand side of Equation (32) and expand it,

$$\begin{aligned} \hat{n} \times [(\hat{n} - \beta) \times \dot{\beta}] &= (\hat{n}\dot{\beta})(\hat{n} - \beta) - (\hat{n}(\hat{n} - \beta))\dot{\beta} \\ &= (\hat{n}\dot{\beta})\hat{n} - \dot{\beta}(\hat{n}\dot{\beta}) - [\dot{\beta} - (\hat{n}\beta)\dot{\beta}] \\ &= -\dot{\beta} - \beta(\hat{n}\dot{\beta}) + (\hat{n}\beta)\dot{\beta} + (\hat{n}\dot{\beta})\hat{n} \end{aligned} \tag{A8}$$

One can see that Equations (A6) and (A8) are equal, and thus we obtain the identity Equation (32).

Appendix C. Derivation of $\Theta^*(\theta, \phi, \chi)$

The velocity vector β is directed along the z^* -axis and the acceleration vector $\dot{\beta}^*$ is lying on the x^*z^* plane. As we mentioned earlier, the angle between velocity and acceleration, as well as the azimuth, will be constant. In the instantaneous rest frame, the angles of θ^* and Θ^* change; therefore we express:

$$\dot{\beta}^* = \dot{\beta}^*(\sin \chi^* \hat{x}^* + \cos \chi^* \hat{z}^*), \tag{A9}$$

and $\beta = \beta \hat{z}^*$. Additionally, we know that the instant unit vector is:

$$\hat{n}^* = \sin \theta^* \cos \phi^* \hat{x}^* + \sin \theta^* \sin \phi^* \hat{y}^* + \cos \theta^* \hat{z}^*. \tag{A10}$$

The dot product between each of the elements gives us:

$$\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta} = \beta \cos \theta^*, \quad (\text{A11})$$

and likewise for the velocity with the instantaneous acceleration vector,

$$\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}^* = \beta \dot{\beta}^* \cos \chi^* \quad (\text{A12})$$

and the unit instant vector with the instant acceleration vector,

$$\hat{\mathbf{n}}^* \cdot \dot{\boldsymbol{\beta}}^* = \dot{\beta}^* \cos \Theta^* = \dot{\beta}^* (\cos \phi^* \sin \theta^* \sin \chi^* + \cos \theta^* \cos \chi^*) \quad (\text{A13})$$

We then rewrite angles of the lab frame in terms of angles of the instantaneous rest frame:

$$\cos \Theta^* = (\cos \phi^* \sin \theta^* \sin \chi^* + \cos \theta^* \cos \chi^*). \quad (\text{A14})$$

Converting from the instant rest frame to the lab frame gives us:

$$\cos \Theta^* = \frac{\cos \phi \sin \theta \sin \chi}{\gamma g} + \frac{(\cos \theta - \beta) \cos \chi}{g} \quad (\text{A15})$$

which is:

$$1 - \sin^2 \Theta^* = \left(\frac{\cos \phi \sin \theta \sin \chi}{\gamma g} + \frac{(\cos \theta - \beta) \cos \chi}{g} \right)^2 \quad (\text{A16})$$

Finally we have the general relationship for $\Theta^*(\theta, \phi, \chi)$:

$$\sin^2 \Theta^* = 1 - \frac{(\cos \phi \sin \theta \sin \chi - \gamma(v - \cos \theta) \cos \chi)^2}{\gamma^2 g^2}. \quad (\text{A17})$$

Equation (A17) explicitly demonstrates how the angle Θ^* in the instantaneous rest frame is related to the lab frame through angles ϕ , θ , and χ .

Appendix D. Limits of Power Distribution

The angular distribution of instantaneous radiated power can be written in two forms, both as Equation (17) and as Equation (23), and despite their different appearance they are equivalent (see also [25]). As a consistency check, we demonstrate their limits in the most well-known cases and confirm they are also the same. First, Equation (23) is expanded and rewritten in terms of lab frame angles θ , ϕ and χ . It is decomposed as follows:

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{16\pi^2} \frac{[\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})]^2}{(1 - \beta \cos \theta)^5} \\ &= \frac{1}{16\pi^2} \frac{[(\hat{\mathbf{n}} - \boldsymbol{\beta})(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})]^2}{(1 - \beta \cos \theta)^5} \\ &= \frac{1}{16\pi^2} \frac{(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - 2\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}} + \beta^2) - 2(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}} - \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}}) + \dot{\boldsymbol{\beta}}^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2}{(1 - \beta \cos \theta)^5} \\ &= \frac{1}{16\pi^2} \frac{(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})^2 (\beta^2 - 1) + 2(\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}) + \dot{\boldsymbol{\beta}}^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2}{(1 - \beta \cos \theta)^5}. \end{aligned} \quad (\text{A18})$$

The coordinate system is chosen so that vector $\boldsymbol{\beta}$ is directed along the z-axis, and the acceleration vector $\dot{\boldsymbol{\beta}}$ lies on the x z plane:

$$\boldsymbol{\beta} = \beta \hat{z}, \quad \dot{\boldsymbol{\beta}} = \dot{\beta} (\sin \chi \hat{x} + \cos \chi \hat{z}), \quad (\text{A19})$$

and the direction of the radiation $\hat{\mathbf{n}}$ is given by:

$$\hat{\mathbf{n}} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}. \quad (\text{A20})$$

Thus, the dot product between the elements \mathbf{n} , $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ gives:

$$\begin{aligned}\hat{\mathbf{n}} \cdot \boldsymbol{\beta} &= \beta \cos \theta, & \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} &= \dot{\beta} \cos \chi, \\ \hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}} &= \dot{\beta} \cos \Theta = \dot{\beta}(\cos \phi \sin \theta \sin \chi + \cos \theta \cos \chi).\end{aligned}\quad (\text{A21})$$

Then, by substituting Equations (A19)–(A21) into the equation of angular distribution in the expanded form Equation (A18), Equation (23) is rewritten as follows:

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2} \frac{\dot{\beta}^2 \cos^2 \Theta (\beta^2 - 1) + 2\dot{\beta}^2 \beta \cos \Theta \cos \chi (1 - \beta \cos \theta) + \dot{\beta}^2 (1 - \beta \cos \theta)^2}{(1 - \beta \cos \theta)^5}. \quad (\text{A22})$$

Now consider the limit of the power distribution, Equation (A22), in the case of parallel and perpendicular positions of the acceleration and velocity vectors. For $\chi = 0$ (i.e., $a \parallel v$), where $\cos \Theta = \cos \theta$:

$$\begin{aligned}\frac{dP_{\parallel}}{d\Omega} &= \frac{1}{16\pi^2} \frac{\dot{\beta}^2 \cos^2 \theta (\beta^2 - 1) + 2\dot{\beta}^2 \beta \cos \theta (1 - \beta \cos \theta) + \dot{\beta}^2 (1 - \beta \cos \theta)^2}{(1 - \beta \cos \theta)^5} \\ &= \frac{1}{16\pi^2} \frac{\dot{\beta}^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5},\end{aligned}\quad (\text{A23})$$

and for $\chi = \pi/2$ (i.e., $a \perp v$), where $\cos \Theta = \cos \phi \sin \theta$:

$$\begin{aligned}\frac{dP_{\perp}}{d\Omega} &= \frac{1}{16\pi^2} \frac{\dot{\beta}^2 \cos^2 \phi \sin^2 \theta (\beta^2 - 1) + \dot{\beta}^2 (1 - \beta \cos \theta)^2}{(1 - \beta \cos \theta)^5} \\ &= \frac{\dot{\beta}^2}{16\pi^2 (1 - \beta \cos \theta)^3} \left(1 - \frac{\cos^2 \phi \sin^2 \theta}{\gamma^2 (1 - \beta \cos \theta)^2} \right).\end{aligned}\quad (\text{A24})$$

The power distribution limits by using Equation (17) in terms of angles are found by applying Equation (A17):

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{16\pi^2} \frac{\gamma^2 [\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2]}{(1 - \beta \cos \theta)^3} \sin^2 \Theta^*(\theta, \phi, \chi) \\ &= \frac{1}{16\pi^2} \frac{\gamma^2 [\dot{\beta}^2 - (\beta \dot{\beta} \sin \chi)^2]}{(1 - \beta \cos \theta)^3} \sin^2 \Theta^*(\theta, \phi, \chi).\end{aligned}\quad (\text{A25})$$

$$\frac{dP}{d\Omega} = \frac{\dot{\beta}^2}{16\pi^2} \frac{\gamma^2 [1 - \beta^2 \sin^2 \chi]}{(1 - \beta \cos \theta)^3} \left(1 - \frac{(\cos \phi \sin \theta \sin \chi - \gamma(\beta - \cos \theta) \cos \chi)^2}{\gamma^2 (1 - \beta \cos \theta)^2} \right). \quad (\text{A26})$$

For $\chi = 0$ (i.e., $a \parallel v$), Equation (A26) transforms to:

$$\frac{dP_{\parallel}}{d\Omega} = \frac{\dot{\beta}^2}{16\pi^2} \frac{\gamma^2}{(1 - \beta \cos \theta)^3} \left(1 - \frac{(\beta - \cos \theta)^2}{(1 - \beta \cos \theta)^2} \right) = \frac{1}{16\pi^2} \frac{\dot{\beta}^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5}, \quad (\text{A27})$$

and for $\chi = \pi/2$ (i.e., $a \perp v$), Equation (A26) converts to:

$$\begin{aligned}\frac{dP_{\perp}}{d\Omega} &= \frac{\dot{\beta}^2}{16\pi^2} \frac{\gamma^2 [1 - \beta^2]}{(1 - \beta \cos \theta)^3} \left(1 - \frac{(\cos \phi \sin \theta)^2}{\gamma^2 (1 - \beta \cos \theta)^2} \right) \\ &= \frac{\dot{\beta}^2}{16\pi^2 (1 - \beta \cos \theta)^3} \left(1 - \frac{\cos^2 \phi \sin^2 \theta}{\gamma^2 (1 - \beta \cos \theta)^2} \right).\end{aligned}\quad (\text{A28})$$

Comparing Equations (A23) and (A27), as well as Equations (A24) and (A28), it can be seen that the limits of Equations (17) and (23) are the same, which gives the correct results for both rectilinear (braking) and circular (cyclotron) distributions.

References

1. DeWitt, B.S. Quantum Field Theory in Curved Space-Time. *Phys. Rept.* **1975**, *19*, 295–357. [CrossRef]
2. Fulling, S.A.; Davies, P.C.W. Radiation from a moving mirror in two dimensional space-time: conformal anomaly. *Proc. R. Soc. Lond. A Math. Phys. Sci.* **1976**, *348*, 393–414.
3. Davies, P.; Fulling, S. Radiation from Moving Mirrors and from Black Holes. *Proc. R. Soc. Lond. A Math. Phys. Sci.* **1977**, *A356*, 237–257. [CrossRef]
4. Ford, L.; Vilenkin, A. Quantum Radiation by Moving Mirrors. *Phys. Rev. D* **1982**, *25*, 2569. [CrossRef]
5. Higuchi, A.; Matsas, G.; Sudarsky, D. Bremsstrahlung and Fulling-Davies-Unruh thermal bath. *Phys. Rev. D* **1992**, *46*, 3450–3457. [CrossRef] [PubMed]
6. Ren, H.; Weinberg, E.J. Radiation from a moving scalar source. *Phys. Rev. D* **1994**, *49*, 6526–6533. [CrossRef]
7. Ritus, V. The Symmetry, inferable from Bogoliubov transformation, between the processes induced by the mirror in two-dimensional and the charge in four-dimensional space-time. *J. Exp. Theor. Phys.* **2003**, *97*, 10–23. [CrossRef]
8. Ritus, V. Vacuum-vacuum amplitudes in the theory of quantum radiation by mirrors in $1 + 1$ -space and charges in $3 + 1$ -space. *Int. J. Mod. Phys. A* **2002**, *17*, 1033–1040. [CrossRef]
9. Ritus, V. Symmetries and causes of the coincidence of the radiation spectra of mirrors and charges in $(1 + 1)$ and $(3 + 1)$ spaces. *J. Exp. Theor. Phys.* **1998**, *87*, 25–34. [CrossRef]
10. Nikishov, A.; Ritus, V. Emission of scalar photons by an accelerated mirror in $(1 + 1)$ space and its relation to the radiation from an electrical charge in classical electrodynamics. *J. Exp. Theor. Phys.* **1995**, *81*, 615–624.
11. Fulling, S.A.; Landulfo, A.G.S.; Matsas, G.E.A. The relation between quantum and classical field theory with a classical source. *Proc. R. Soc. Lond. A Math. Phys. Sci.* **2020**, *476*, 20200656.
12. Landulfo, A.G.; Fulling, S.A.; Matsas, G.E. Classical and quantum aspects of the radiation emitted by a uniformly accelerated charge: Larmor-Unruh reconciliation and zero-frequency Rindler modes. *Phys. Rev. D* **2019**, *100*, 045020. [CrossRef]
13. Cozzella, G.; Fulling, S.A.; Landulfo, A.G.; Matsas, G.E. Uniformly accelerated classical sources as limits of Unruh-DeWitt detectors. *Phys. Rev. D* **2020**, *102*, 105016. [CrossRef]
14. Unruh, W.G. Notes on black-hole evaporation. *Phys. Rev. D* **1976**, *14*, 870–892. [CrossRef]
15. Birrell, N.; Davies, P. *Quantum Fields in Curved Space*; Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press: Cambridge, UK, 1984. [CrossRef]
16. Lin, K.N.; Chou, C.E.; Chen, P. Particle Production by a Relativistic Semi-Transparent Mirror in $1 + 3$ D Minkowski Spacetime. *arXiv* **2020**, arXiv:2008.12251.
17. Chen, P.; Mourou, G. Accelerating Plasma Mirrors to Investigate Black Hole Information Loss Paradox. *Phys. Rev. Lett.* **2017**, *118*, 045001. [CrossRef] [PubMed]
18. Chen, P.; Mourou, G. Trajectory of a flying plasma mirror traversing a target with density gradient. *Phys. Plasmas* **2020**, *27*, 123106. [CrossRef]
19. Brown, E.; Louko, J. Smooth and sharp creation of a Dirichlet wall in $1 + 1$ quantum field theory: how singular is the sharp creation limit? *J. High Energ. Phys.* **2015**, *61*, 2015. [CrossRef]
20. Dodonov, V. Fifty Years of the Dynamical Casimir Effect. *Physics* **2020**, *2*, 67–104. [CrossRef]
21. Kosyakov, B.P. *Introduction to the Classical Theory of Particles and Fields*; Springer: Berlin/Heidelberg, Germany, 2007. [CrossRef]
22. Kosyakov, B.P. Self-interaction in classical gauge theories and gravitation. *Phys. Rept.* **2019**, *812*, 1–55. [CrossRef]
23. Good, M.R.; Temir Khan, M.; Oikonomou, T. Stationary Worldline Power Distributions. *Int. J. Theor. Phys.* **2019**, *58*, 2942–2968. [CrossRef]
24. Good, M.; Juárez-Aubry, B.A.; Moustos, D.; Temir Khan, M. Unruh-like effects: Effective temperatures along stationary worldlines. *JHEP* **2020**, *06*, 059. [CrossRef]
25. McDonald, K.T. Radiated-Power Distribution in the Far Zone of a Moving System, (Unpublished) 2017. Available online: http://kirkmcd.princeton.edu/examples/moving_far.pdf (accessed on 28 April 2017).
26. Rohrlich, F. *Classical Charged Particles*, 3rd ed.; World Scientific: New Jersey, NJ, USA, 2007.
27. Walker, W.R. Particle and energy creation by moving mirrors. *Phys. Rev. D* **1985**, *31*, 767–774. [CrossRef] [PubMed]
28. Fabbri, A.; Navarro-Salas, J. *Modeling Black Hole Evaporation*; Imperial College Press: London, UK, 2005.
29. Good, M.R.R.; Anderson, P.R.; Evans, C.R. Time dependence of particle creation from accelerating mirrors. *Phys. Rev. D* **2013**, *88*, 025023. [CrossRef]
30. Bekenstein, J.D.; Mayo, A.E. Black holes are one-dimensional. *Gen. Rel. Grav.* **2001**, *33*, 2095–2099. [CrossRef]
31. Rindler, W.A. *Essential Relativity: Special, General and Cosmological*, 2nd ed.; Texts and Monographs in Physics; Springer: New York, NY, USA, 1977. [CrossRef]
32. Rahaman, F. *The Special Theory of Relativity: A Mathematical Approach*; Springer: New Delhi, India, 2014. [CrossRef]
33. Jackson, J.D. *Classical Electrodynamics*, 3rd ed.; Wiley: New York, NY, USA, 1999.
34. Zangwill, A. *Modern Electrodynamics*; Cambridge Univ. Press: Cambridge, UK, 2013.
35. Griffiths, D.J. *Introduction to Electrodynamics*, 4th ed.; Re-published by Cambridge University Press in 2017; Pearson: Boston, MA, USA, 2013. [CrossRef]
36. Parker, L. Gravitational Particle Production in the Formation of Cosmic Strings. *Phys. Rev. Lett.* **1987**, *59*, 1369–1372. [CrossRef]
37. Fulton, T.; Rohrlich, F. Classical radiation from a uniformly accelerated charge. *Ann. Phys.* **1960**, *9*, 499–517. [CrossRef]

38. Boulware, D.G. Radiation from a uniformly accelerated charge. *Ann. Phys.* **1980**, *124*, 169–188. [[CrossRef](#)]
39. Singal, A.K. Discrepancy between power radiated and the power loss due to radiation reaction for an accelerated charge. *Symmetry* **2020**, *12*, 1833, [[CrossRef](#)]
40. Kay, B.S.; Lupo, U. Non-existence of isometry-invariant Hadamard states for a Kruskal black hole in a box and for massless fields on 1 + 1 Minkowski spacetime with a uniformly accelerating mirror. *Class. Quant. Grav.* **2016**, *33*, 215001, [[CrossRef](#)]
41. Obadia, N.; Parentani, R. Uniformly accelerated mirrors. Part 1. Mean fluxes. *Phys. Rev. D* **2003**, *67*, 024021, [[CrossRef](#)]
42. Obadia, N.; Parentani, R. Uniformly accelerated mirrors. Part 2. Quantum correlations. *Phys. Rev. D* **2003**, *67*, 024022, [[CrossRef](#)]
43. Lee, J.S.; Cleaver, G.B. The relativistic blackbody spectrum in inertial and non-inertial reference frames. *New Astron.* **2017**, *52*, 20–28. [[CrossRef](#)]
44. Einstein, A. On the relativity principle and the conclusions drawn from it. *Jahrb. Radioakt. Elektron* **1907**, *4*, 411–462.
45. Planck, M. Zur dynamik bewegter systeme. *Ann. Phys. (Liepzig)* **1908**, *26*, 1.
46. Ott, H. Lorentz-Transformation der Wärme und der Temperatur. *Z. Phys.* **1963**, *175*, 70. [[CrossRef](#)]
47. Landsberg, P. Does a Moving Body appear Cool? *Nature* **1967**, *214*, 903. [[CrossRef](#)]
48. Carlitz, R.D.; Willey, R.S. Reflections on moving mirrors. *Phys. Rev. D* **1987**, *36*, 2327–2335. [[CrossRef](#)] [[PubMed](#)]
49. Good, M.R.R.; Anderson, P.R.; Evans, C.R. Mirror Reflections of a Black Hole. *Phys. Rev. D* **2016**, *94*, 065010. [[CrossRef](#)]
50. Good, M.R.; Ong, Y.C. Particle spectrum of the Reissner–Nordström black hole. *Eur. Phys. J. C* **2020**, *80*, 1169. [[CrossRef](#)]
51. Foo, J.; Good, M.R.; Mann, R.B. Particle production model for general classes of Taub-NUT black holes. *arXiv* **2020**, arXiv:2012.02348.
52. Good, M.R.; Foo, J.; Linder, E.V. Accelerating boundary analog of a Kerr black hole. *arXiv* **2020**, arXiv:2006.01349.
53. Good, M.R.; Zhakenuly, A.; Linder, E.V. Mirror at the edge of the universe: Reflections on an accelerated boundary correspondence with de Sitter cosmology. *Phys. Rev. D* **2020**, *102*, 045020. [[CrossRef](#)]
54. Rothman, T. Nonthermal nature of extremal Kerr black holes. *Phys. Lett. A* **2000**, *273*, 303–309. [[CrossRef](#)]
55. Good, M.R. Extremal Hawking radiation. *Phys. Rev. D* **2020**, *101*, 104050. [[CrossRef](#)]
56. Liberati, S.; Rothman, T.; Sonogo, S. Nonthermal nature of incipient extremal black holes. *Phys. Rev. D* **2000**, *62*, 024005. [[CrossRef](#)]