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Strong coupling expansion of the entanglement entropy of Yang-Mills gauge theories

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Abstract

We propose a novel prescription for calculating the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$, where g is the coupling constant. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the n-sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term is negative, which could in principle be canceled by taking into account the "cosmological constant" living in interface of the two entangled sub-regions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We further speculate this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

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1. Introduction

Entanglement entropy is a measure of the level of entanglement between the degrees of freedom in two subregions of a physical system. Besides being a fundamental and mysterious property of quantum mechanics and quantum field theory, quantum entanglement is of practical use. In some systems, the entanglement entropy plays the role of an order parameter characterizing the quantum phase transitions, and demonstrates the scaling behavior near the critical point [1–6]. In field theory, a widely used method for calculating the entanglement entropy is the replica trick [7]. This method calculates the trace of the reduced density matrix to the n-th power in the path integral formalism, which amounts to computing the free energy of the system on a n-sheeted Riemann surface, or equivalently on a "cone" with a conical angle of $2n\pi$. The entanglement entropy is then obtained as a response of the free energy to the change of the conical angle at $n = 1$. This is similar to calculating the black hole entropy by evaluating the response of the free energy of fields with respect to the deficit angle in the Euclidean spacetime [8].

In previous studies, while the computation of entanglement entropy for the scalar and the spinor fields are considered straightforward, it becomes more subtle for the gauge fields. It was found by Kabat [8] that the gauge fields in the black hole entropy method yields an extra negative contact term compared to their entanglement entropy. This term is due to the interaction of the particles with the horizon, and is believed to be related to the choice of the boundary condition while removing the tip of the cone due to the black hole in the Euclidean space. It was later interpreted by [9] as arising from incorrect treatment of the zero modes.

The ambiguity in calculating the entanglement entropy of the gauge fields is caused by that the global Hilbert space of gauge invariant physical state does not admit a direct product decomposition between the regions on the two sides of the entangling surface. To deal with this issue, [10–12] propose to embed the original Hilbert space in an extended Hilber space allowing a tensor product factorization.

On the other hand, the entanglement entropy of the gauge theories had been studied with the Hamiltonian [11,13–18] and the Lagrangian [19–22] approaches. In the Hamiltonian approach for the case of the gauge fields on the lattice [13–16], one needs to impose the Gauss law or the gauge fixing constraints in order to get rid of the unphysical degrees of freedom. This leads to the difficulty in decomposing the global gauge invariant states into the direct products of those living in each subregion. It was proposed [13] that these ambiguities might be compensated by edge modes living in the interface of the subregions and determined by the transverse electric fields. This echoes the concept that the contact term in [8] arises from the sources on the horizon. In [13] this negative contact term was found to arise from the entanglement of the edge modes. Moreover, in order to clarify the issue of the Hilbert space decomposition, [15] employs various choices of the electric, the magnetic, and the trivial centers in the operator algebra in the Hamiltonian formalism. The electric and magnetic choices also result in different tripartite topological entanglement entropy. By using Hamiltonian formalism dominated by the electric term, [18] shows that for the $SU(N)$ lattice gauge theory in the strongly coupled regime, the entanglement entropy behaves as $|\partial V_\perp| \frac{\log g^2 N}{g^8 N^2}$.

In view of these ambiguities, we use a complementary approach – the Lagrangian formalism – to shed light on the entanglement entropy of the Yang-Mills theory on the lattice from a different perspective. We apply the replica method and take the derivative of the trace of n copies of the reduce density matrix by n. Since the density matrix[1] can always be expressed in the path integral formalism, the replica method is valid for the gauge fields at continuum. We start from the replica method at continuum, and then discretize the spacetime into a hypercubic lattice. We divide an infinitely large system into two semi-infinite subregions by a flat boundary, and decompose the spacetime into a direct product of a 2-dimensional cone with a conical angle of $2n\pi$ (or equivalently an n-sheeted Riemann surface) and an ordinary Euclidean space transverse to the cone. (See Section 3 for the detail.) Then we use the Wilson gauge action on a discrete lattice. This action sums over the Wilson loops on the plaquettes, including those on the n-sheeted Riemann surface and those on the ordinary Euclidean space. In contrast with the previous studies where the conical singularity is placed on the lattice site, we use a different discretization setup by locating the conical singularity in the center of the plaquette. As a result, the branch lines on the n-sheeted Riemann surface cut across the links, and there is no lattice site on the cut. (See Fig. 1.) This setup yields two types of plaquettes on the lattice of the cone: the central plaquettes encircling the tip of the cone, formed by $4n$ links, and the regular plaquettes formed by 4 links with no singularity inside. When the conical angle, or n, changes, only the central plaquettes (i.e. those plaquettes with $4n$ links) respond to this change. As a result, entanglement entropy necessarily involves those central plaquettes. The fact that all of the central plaquettes live across the interface between the two subregions naturally give rise to the area law, which states that the leading contribution to entanglement entropy scales as the area of the interface.

The connection to the area law can be further demonstrated order by order diagrammatically under the strong coupling expansion (see for example [23]). Interestingly, we find the leading term in the strong coupling expansion negative. However, symmetries of the action allow a two-dimensional cosmological constant living in the interface [24] which could provide a positive contribution at an even lower order. We speculate that the freedom to tune this unknown 2-dimensional cosmological constant corresponds to the ambiguities encountered in the Hamiltonian approach. We further speculate that the 2-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. They can play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

This paper is organized as follows. In Sec. 2, we briefly review the notion of entanglement entropy and the replica method. The entanglement entropy of Yang-Mills fields on the lattice under the strong coupling expansion is calculated in Sec. 3, and the cancellation of the negative contribution by including a 2-dimensional cosmological constant is discussed in Sec. 4. Sec. 5 compares our result with the previous ones obtained by the Hamiltonian methods, and Sec. 6 summarizes our study.

2. Entanglement entropy and the replica method

Suppose our system occupies an infinitely large and flat $d+1=4$ dimensional spacetime. The $d=3$ dimensional space is divided into two regions A and B by a time independent, infinite, and flat 2-dimensional space-like boundary. The entanglement entropy (EE) of a quantum theory between the two subregions is defined by the von Neumann entropy. With some simple algebra, it can be re-expressed as:

[1] We remind the readers that, here and in the following, by "the density matrix" we mean "the ground state density matrix at strong coupling".

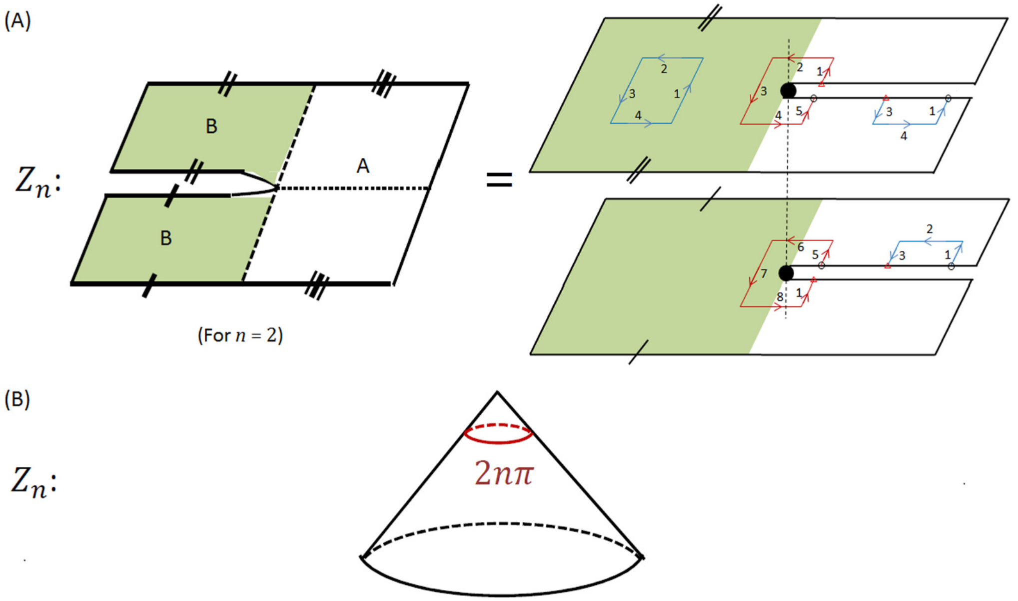

Fig. 1. The n-sheeted Riemann surface in the replica trick. (A) illustrates the equivalent geometric structures of the n-sheeted manifold in the case of $n = 2$ arizing from the replica trick, on which the partition function Z_n is computed. The unshaded and the shaded parts represent the subregion A and B, respectively. The subregion B is traced out in the reduced density matrix ρ_A. The 2 sheets on the right are on top of each other at the same $x_\perp$ coordinates, attaching to each other at the conical singularity across the branch cut. The conical sigularity is represented by the black dot and the dotted vertical line. In the right figure, it is also demonstrated explicitly that, after discretization, the central plaquette with $4n$ edges encircling the conical singularity, and the ordinary plaquettes with 4 links located on a sheet or across two adjacent sheets (see also Fig. 3). The numbers label the order of the link variables forming the plaquettes. As the sizes of the space and the Euclidean time are both infinite (i.e. $T = 0$), this geometry is equivalent to a cone with $2n\pi$ conical angle depicted in (B).

$$S_{\rm EE} = -{\rm tr}[\rho_A \ln \rho_A] = -\left.\frac{\partial}{\partial n}\right|_{n\to 1} \ln {\rm tr}[\rho_A^n] \tag{1}$$

where $\rho_A = {\rm tr}_B[\rho]$ is the reduced density matrix by tracing out the degrees of freedom in region B. This expression is called the replica method because it involves n copies of ρ_A.

An elegant path integral formulation to compute the entanglement entropy using the replica method was first introduced in [7] (see also [25]). In this set up, one recalls that $\rho_{ij} = \langle i \left| e^{-H/T} \right| j \rangle$ and Tr[ρ] is the partition function calculated in finite temperature field theory with appropriate boundary conditions (periodic and anti-periodic boundary conditions for bosons and fermions, respectively) imposed for fields at Euclidean time $\tau = 0$ and $1/T$, where T is the temperature. Then Tr[ρ^2] can be computed by doubling the period (by imposing appropriate boundary conditions at $\tau = 0$ to $2/T$). Similarly, Tr[ρ_A^2] is computed by doubling the period (from 0 to $2/T$) for region A while maintaining the single period (from 0 to $1/T$) for the region B, as shown in the left plot of Fig. 1(A), which is equivalent to performing the path integral on a 2-sheeted Riemann surface in the right plot. One can generalize this set up to Tr[ρ_A^n] for an arbitrary n. There is no restriction on the space partition between A and B. The sizes of A, B, and T can be either finite or infinite.

In this paper, we will just concentrate on the simplest case with the sizes of space and (Euclidean) time to be both infinite (i.e. $T = 0$) and the interface between A and B to be a flat infinite

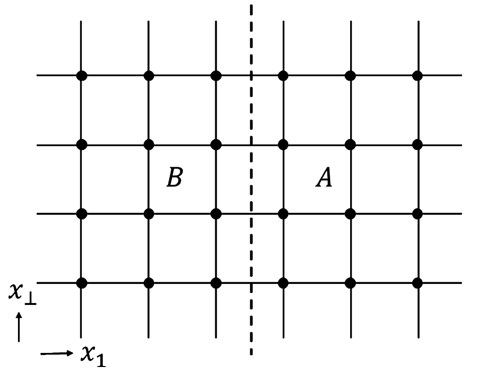

Fig. 2. The cartoon diagram depicting our system setup in which the boundary between the regions A and B cuts across the links connecting the two regions. In our coordinates, x_1 is orthogonal to the boundary, while the boundary is along the (x_2, x_3) dimensions which are abbreviated as $x_\perp$. In this diagram, one of the $x_\perp$ dimensions is suppressed.

plane. In this limit, the n-sheeted Riemann surface has a conical structure as shown in Fig. 1(B) with the time and longitudinal spatial direction (orthogonal to the interface between the A, B region) lying on the cone while the space on the interface transverse to the cone.

As a result, $\mathrm{tr}[\rho_A^n]$ becomes a partition function Z_n on the n-sheeted Riemann surface, or, in our case, a cone with $2n\pi$ conical angle, normalized by the n-copies of the partition function on the ordinary Euclidean space, Z_1^n:

$$\mathrm{tr}[\rho_A^n] = \frac{Z_n}{Z_1^n}, \tag{2}$$

which ensures that as $n = 1$, $\mathrm{tr}[\rho_A^n] = 1$. The entanglement entropy is then given by

$$S_{\mathrm{EE}} = -\left.\frac{\partial}{\partial n}\left(\ln Z_n - n \ln Z_1\right)\right|_{n\to 1} \overset{n=1+\epsilon}{=} -\frac{1}{\epsilon}\left[\ln Z_{1+\epsilon} - (1+\epsilon)\ln Z_1\right]_{\epsilon\to 0}. \tag{3}$$

Note that in the integral of Z_n, n is taken as an integer. After the analytic expression for $\mathrm{tr}[\rho^n]$ is obtained, then n can be analytically extended to non-integers to carry out the differentiation at $n = 1$.

3. Calculation on an n-sheeted lattice manifold

We now discretize the 4-dimensional spacetime by a hypercubic lattice. The 3 spatial dimensions are labeled by the coordinates (x_1, x_2, x_3), while the Euclidean time direction is labeled by τ. Our system is set up such that the flat boundary between the regions A and B cuts across the links connecting the lattice sites in the two regions, as shown in Fig. 2. The boundary is in the (x_2, x_3) plane located at $x_1 = 0$.

When we calculate Z_n, the spacetime is decomposed into the direct product of a $1 + 1$ dimensional n-sheeted Riemann lattice and an Euclidean squared lattice spanning the other 2 dimensions. The $1 + 1$ dimensional n-sheeted Riemann surface is labeled by (x_1, τ), while the other two Euclidean dimensions are described by the coordinates (x_2, x_3). In the following sections in this paper, for our convenience we will call x_1 by $x_\parallel$, and (x_2, x_3) by $x_\perp$. We remind the readers that $\parallel$ and $\perp$ directions are taken with respect to the (x_1, τ) Riemann surface, but not to

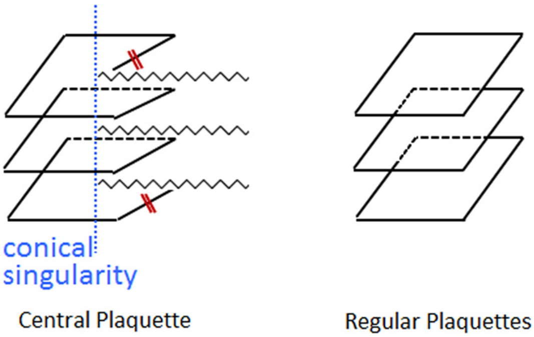

Fig. 3. One central plaquette (left) enclosing the conical singularity v.s. three regular plaquettes on three different sheets (right) in the case of $n = 3$ sheets. In the left figure, the dotted line represents the conical singularity to which all the three sheets attach at the same $x_\perp$., and the wavy lines are cuts on each sheet. The central plaquette has 12 edges and it takes 6π to go around it, while each regular plaquette has 4 links.

the boundary surface of the regions A, B. In the circumstances that using the boundary as the reference plane is necessary, we will describe explicitly in the text.

By means of this discritization, the end point of the branch cut on the n-sheeted Riemann surface (corresponding to the conical singularity at the tip the cone) locates inside a plaquette, rather than sitting on a lattice site. As a result of the discretization, there are two types of plaquettes. The *ordinary plaquettes* contain no conical singularity inside, and are composed of 4 links. They are in the transverse $x_\perp$ and the parallel $(x_\parallel, \tau)$ dimensions. Their plaquette variables are denoted by $U_\square^{(k)} = \mathcal{P}\prod_{l\in\square} U_l^{(k)}$, where k indicates that they locate on the k-th sheet, and $\mathcal{P}$ signifies the ordered product of the link variables $U_l^{(k)}$ forming the plaquette. On the other hand, the *central plaquette* going around all the n sheets encloses the conical singularity on the Riemann surface in the parallel $(x_\parallel, \tau)$ dimensions, and are composed of $4n$ links. Their plaquette variables are expressed by $U_{\boxdot}(x_\perp) = \mathcal{P}\prod_{l\in\boxdot} U_l$, where $x_\perp$ is the location of the $(x_\parallel, \tau)$ planes in the transverse dimensions. See Figs. 1 and 3 for the illustration of these two types of plaquettes. It will be clear later that our result is independent of the location of the conical singularity as long as it is encircled by a plaquette. For simplicity we choose that the conical singularity sits in the center of a plaquette.

Recall that the partition function of the lattice gauge theory on a one-sheet manifold is given by

$$Z = \int \mathcal{D}U \exp\left\{-\beta \sum_{\square}\left[1 - \frac{1}{N}\mathrm{Re}\,\mathrm{tr}U_\square\right]\right\} \underset{a\to 0}{\longrightarrow} \int \mathcal{D}A \exp\left\{-\int d^4x \left[\frac{1}{4g^2}\mathrm{tr}F^2\right]\right\}, \tag{4}$$

where $\beta = 2N/g^2$ and $\square$ labels the location of plaquettes. The plaquette variable $U_\square$ is the local Wilson loop composed of the ordered product of four gauge links, $U_\square = \mathcal{P}\prod_{l\in\square} U_l$ where $\mathcal{P}$ indicates the ordered product and U_l is the link variable representing the gauge fields. The action recovers the Yang-Mills action in the continuum limit by setting the lattice spacing $a \to 0$. We assume that the lattice has the same spacing a in all dimensions.

To construct a lattice gauge field system in the $d + 1 = 4$ dimensions whose $1 + 1$ dimensions is an n-sheeted manifold, we rewrite the partition function in Eq. (4) in terms of the ordinary plaquettes $U_\square^{(k)}$ and the central ones $U_{\boxdot}(x_\perp)$, such that the partition function reads

$$
\begin{aligned}
Z_n &= \int \left[\prod_{m=1}^{n} \mathcal{D}U^{(m)}\right] \exp\left\{\frac{\beta}{N}\sum_{k=1}^{n}\sum_{\square}\mathrm{Retr}[U_{\square}^{(k)}-1]+\frac{\beta}{nN}\sum_{\mathbf{x}_\perp}\mathrm{Re\,tr}[U_{\boxdot}(\mathbf{x}_\perp)-1]\right\} \qquad (5)\\
&\underset{a\to 0}{\to} \int \left[\prod_{m=1}^{n} \mathcal{D}A^{(m)}\right] \exp\left\{-\frac{1}{4g^2}\int d^2x_\perp\Big(\sum_{k=1}^{n}\int_{\mathbb{R}^2-\{0\}} d^2x_\parallel \mathrm{tr}F^{(k)2} \right.\\
&\qquad\left. +\frac{1}{n}\sum_{k,l=1}^{n}\int_{\{0\}} d^2x_\parallel \mathrm{tr}\left[F^{(k)}F^{(l)}\right]\Big)\right\}\\
&= \int \left[\prod_{m=1}^{n} \mathcal{D}A^{(m)}\right] \exp\left\{-\frac{1}{4g^2}\sum_{k=1}^{n}\int d^2x_\perp \int d^2x_\parallel \mathrm{tr}F^{(k)2}\right\},
\end{aligned}
$$

where the upper indices (k) label different sheets on the Riemann surface. We have assumed the boundary condition

$$
F^{(k)}(\mathbf{x}_\perp,\mathbf{x}_\parallel=\mathbf{0}) = F^{(l)}(\mathbf{x}_\perp,\mathbf{x}_\parallel=\mathbf{0}),\ (k\neq l), \tag{6}
$$

i.e. the field strength on each sheet is assumed be equal. This condition is natural if we impose 2π rotation symmetry on the n-sheeted surface. We have introduced an extra $1/n$ factor to the central plaquette terms, because the $4n$ gauge links of the central plaquettes encircle the F_{01} flux over an area na^2, and give rise to a factor $\propto n^2a^4F_{01}^2$ to the action, inconsistent with the contribution from the transverse plaquettes scaling as na^4. Therefore a factor of $1/n$ is introduced to compensate this effect.

Expanding Eq. (5) to the second order of β, we have

$$
\begin{aligned}
Z_n = e^{-\beta n\mathcal{N}-\frac{\beta}{n}\mathcal{N}_\perp} \int \left[\prod_{m=1}^{n} \mathcal{D}U^{(m)}\right] &\left\{1+\frac{\beta}{N}\sum_{k=1}^{n}\sum_{\square}\mathrm{Re\,tr}\left[U_{\square}^{(k)}\right]\right.\\
&+\frac{\beta}{nN}\sum_{\mathbf{x}_\perp}\mathrm{Re\,tr}\left[U_{\boxdot}(\mathbf{x}_\perp)\right]+\frac{\beta^2}{2N^2}\sum_{k,l=1}^{n}\sum_{\square,\square'}\left[\mathrm{Re\,tr}U_{\square}^{(k)}\right]\left[\mathrm{Re\,tr}U_{\square'}^{(l)}\right]\\
&+\frac{\beta^2}{2N^2}\frac{1}{n^2}\sum_{\mathbf{x}_{\perp1},\mathbf{x}_{\perp2}}\left[\mathrm{Re\,tr}U_{\boxdot}(\mathbf{x}_{\perp1})\right]\left[\mathrm{Re\,tr}U_{\boxdot}(\mathbf{x}_{\perp2})\right]\\
&+\frac{\beta^N}{N!N^N}\left[\sum_{k=1}^{n}\sum_{\square}\mathrm{Re\,tr}\left[U_{\square}^{(k)}\right]+\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re\,tr}\left[U_{\boxdot}(\mathbf{x}_\perp)\right]\right]^N\\
&\left.+\beta^2(\mathrm{tr}U_{\square})(\mathrm{tr}U_{\boxdot})\text{ cross-terms}+\text{other higher order terms in}\beta\right\}.
\end{aligned} \tag{7}
$$

In this equation, N is the number of colors. $\mathcal{N}$ is the number of plaquettes not encircling the conical singularity on a single sheet, including those parallel and transverse to the $(x_\parallel,\tau)$ Riemann surface. $\mathcal{N}_\perp$ is the number of central plaquettes encircling the conical singularity at $(x_\parallel=0,\tau=0)$ if we count across all $x_\perp$ coordinates, i.e. $\mathcal{N}_\perp = A_\perp/a^2$ where $A_\perp$ is the area of the boundary between the regions A, B and a is the lattice size. We isolate the $O(\beta^N)$ terms in the third line of (7) explicitly from "other higher order terms in β", because they will contribute to $O(\beta^2)$ and $O(\beta^3)$ in the $N = 2, 3$ cases respectively. It will be understood more clearly in the following.

To calculate the result, we apply Re $\mathrm{tr}U = \frac{1}{2}(\mathrm{tr}U + \mathrm{tr}U^\dagger)$ to (7), and use the Harr measure

$$\int dU U_{ij} = 0 = \int dU U^\dagger_{ij}, \tag{8}$$

$$\int dU U_{ij} U^\dagger_{kl} = \frac{1}{N}\delta_{il}\delta_{jk}, \tag{9}$$

such that

$$\int \mathcal{D}U \mathrm{tr}U^{(k)}_{\square} \mathrm{tr}U^{(l)\dagger}_{\square} = \delta_{kl}, \tag{10}$$

$$\int \mathcal{D}U \ \mathrm{tr}U_{\boxdot}(\mathbf{x}_{\perp 1}) \ \mathrm{tr}U^\dagger_{\boxdot}(\mathbf{x}_{\perp 2}) = \delta_{x_{\perp 1}x_{\perp 2}}, \tag{11}$$

to compute the contribution of the Wilson loops from $\int \mathcal{D}U \ \mathrm{tr}U\mathrm{tr}U^\dagger$. Note that (10) and (11) are independent of n, and $\mathcal{D}U$ in (10) and (11) are for integration of all link variables. It can be derived that the $(\mathrm{tr}U_{\square})(\mathrm{tr}U_{\boxdot})$ cross-terms vanishes in (7). We then obtain

$$\begin{aligned}
Z_n = e^{-\beta n\mathcal{N} - \frac{\beta}{n}\mathcal{N}_\perp} \Bigg\{ & 1 + \frac{1}{2}\frac{\beta^2}{2N^2}\sum_{k,l=1}^{n}\sum_{\square,\square'}\int\left[\prod_{m=1}^{n}\mathcal{D}U^{(m)}\right]\left[\mathrm{tr}U^{(k)}_{\square}\right]\left[\mathrm{tr}U^{(l)\dagger}_{\square'}\right] \\
& + \frac{1}{2}\frac{\beta^2}{2N^2}\frac{1}{n^2}\sum_{\mathbf{x}_{\perp 1},\mathbf{x}_{\perp 2}}\int\left[\prod_{k=1}^{n}\mathcal{D}U^{(k)}\right]\left[\mathrm{tr}U_{\boxdot}(\mathbf{x}_{\perp 1})\right]\left[\mathrm{tr}U^\dagger_{\boxdot}(\mathbf{x}_{\perp 2})\right] \\
& + \frac{\beta^N}{N!N^N}\int\left[\prod_{m=1}^{n}\mathcal{D}U^{(m)}\right]\left[\left(\sum_{k=1}^{n}\sum_{\square}\mathrm{Re}\ \mathrm{tr}\left[U^{(k)}_{\square}\right]\right)^N \right. \\
& \left. + \left(\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re}\ \mathrm{tr}\left[U_{\boxdot}(\mathbf{x}_\perp)\right]\right)^N\right] + O(\beta^4)\Bigg\} \\
= e^{-\beta n\mathcal{N} - \frac{\beta}{n}\mathcal{N}_\perp} & \left\{1 + \frac{\beta^2}{4N^2}n\mathcal{N} + \frac{\beta^2}{4N^2}\frac{1}{n^2}\mathcal{N}_\perp + \frac{2}{N!}\left(\frac{\beta}{2N}\right)^N\left[n\mathcal{N} + \frac{\mathcal{N}_\perp}{n^N}\right] + O(\beta^4)\right\}.
\end{aligned} \tag{12}$$

Because of the Harr measure (10) and (11), the $O(\beta^3)$ terms in the "other higher order terms in β" of (7) vanish. They instead start from $O(\beta^4)$ order, corresponding to the $\mathrm{tr}U_{\square}\mathrm{tr}U^\dagger_{\square}\mathrm{tr}U_{\square}\mathrm{tr}U^\dagger_{\square}$ and $\mathrm{tr}U_{\boxdot}\mathrm{tr}U^\dagger_{\boxdot}\mathrm{tr}U_{\boxdot}\mathrm{tr}U^\dagger_{\boxdot}$ terms.

The N Wilson loops interactions the third line of (12) is derived from the third line of (7), and give rise to the β^N terms in the result of (12). We explain how to calculate these terms as follows. By making use of the identities

$$\int dU U^{(p)}_{i_1 j_1} U^{(q)}_{i_2 j_2} ... U^{(s)}_{i_N j_N} = \frac{1}{N!}\epsilon_{i_1 i_2 ... i_N}\epsilon_{j_1 j_2 ... j_N}\delta^{pq\cdots s}, \tag{13}$$

$$\int dU U^{(p)\dagger}_{i_1 j_1} U^{(q)\dagger}_{i_2 j_2} ... U^{(s)\dagger}_{i_N j_N} = \frac{1}{N!}\epsilon_{i_1 i_2 ... i_N}\epsilon_{j_1 j_2 ... j_N}\delta^{pq\cdots s}, \tag{14}$$

the contribution from the N Wilson loops interactions $\int \mathcal{D}U\,(\mathrm{tr}U)^N$ and $\int \mathcal{D}U\left(\mathrm{tr}U^\dagger\right)^N$ of $O(\beta^N)$ are

$$\int \mathcal{D}U\,(\mathrm{tr}U_{\square})^N = \int \mathcal{D}U\left(\mathrm{tr}U^{\dagger}_{\square}\right)^N = 1, \tag{15}$$

$$\int \mathcal{D}U\,[\mathrm{tr}U_{\boxdot}(\mathbf{x}_\perp)]^N = \int \mathcal{D}U\left[\mathrm{tr}U^{\dagger}_{\boxdot}(\mathbf{x}_\perp)\right]^N = 1. \tag{16}$$

Therefore, the $O(\beta^N)$ contribution from these terms can be computed,

$$\begin{aligned}
Z_n(\beta^N) &= \int\left[\prod_{m=1}^n \mathcal{D}U^{(m)}\right]\frac{\beta^N}{N!N^N}\left\{\sum_{k=1}^n\sum_{\square}\mathrm{Re}\,\mathrm{tr}\left[U^{(k)}_{\square}\right]+\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re}\,\mathrm{tr}\,[U_{\boxdot}(\mathbf{x}_\perp)]\right\}^N \\
&= \int\left[\prod_{m=1}^n \mathcal{D}U^{(m)}\right]\frac{\beta^N}{N!N^N}\left\{\left(\sum_{k=1}^n\sum_{\square}\mathrm{Re}\,\mathrm{tr}\left[U^{(k)}_{\square}\right]\right)^N\right. \\
&\quad\left.+\left(\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re}\,\mathrm{tr}\,[U_{\boxdot}(\mathbf{x}_\perp)]\right)^N\right\} \\
&= \int\left[\prod_{m=1}^n \mathcal{D}U^{(m)}\right]\frac{\beta^N}{N!(2N)^N}\left\{\left(\sum_{k=1}^n\sum_{\square}\mathrm{tr}\left[U^{(k)}_{\square}\right]\right)^N+\left(\sum_{k=1}^n\sum_{\square}\mathrm{tr}\left[U^{(k)\dagger}_{\square}\right]\right)^N\right. \\
&\quad\left.+\left(\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re}\,\mathrm{tr}\,[U_{\boxdot}(\mathbf{x}_\perp)]\right)^N+\left(\frac{1}{n}\sum_{\mathbf{x}_\perp}\mathrm{Re}\,\mathrm{tr}\left[U^{\dagger}_{\boxdot}(\mathbf{x}_\perp)\right]\right)^N+\text{cross-terms}\right\} \\
&= \frac{2}{N!}\left(\frac{\beta}{2N}\right)^N\left\{n\mathcal{N}+\frac{\mathcal{N}_\perp}{n^N}+\text{cross-terms}\right\}.
\end{aligned} \tag{17}$$

This explains how the β^N terms in (12) appear. They become subleading for $N \geq 3$.

With (12), we can take the combination

$$\ln Z_n - n\ln Z_1 = -\beta\mathcal{N}_\perp\left[\frac{1}{n}-n\right]+\frac{\beta^2\mathcal{N}_\perp}{4N^2}\left(\frac{1}{n^2}-n\right)+\frac{2}{N!}\left(\frac{\beta}{2N}\right)^N\left[n\mathcal{N}+\frac{\mathcal{N}_\perp}{n^N}\right] +O(\beta^4), \tag{18}$$

and then obtain the Renyi entropy

$$\begin{aligned}
S_n &= \frac{\ln Z_n - n\ln Z_1}{1-n} \\
&= -\beta\mathcal{N}_\perp\left(\frac{1+n}{n}\right)+\frac{\beta^2\mathcal{N}_\perp}{4N^2}\left(\frac{1+n+n^2}{n^2}\right)+\frac{2}{N!}\left(\frac{\beta}{2N}\right)^N\mathcal{N}_\perp\left(\frac{1}{n^N}-n\right)\frac{1}{1-n} \\
&\quad +O(\beta^4).
\end{aligned} \tag{19}$$

This result implies, no matter how we place the lattice, as long as the conical singularity is located inside the central plaquettes (i.e. the central plaquettes exist), the result will be the same. But if one chooses to put the conical singularity on a lattice site, then there is no central plaquettes to offer the $\frac{1}{n}$ and $\frac{1}{n^2}$ factor in (18), but the factors n appears instead, and the outcome of $\ln Z_n$ is canceled by that from $n\ln Z_1$, and the Renyi entropy will vanish.

We express the Renyi entropy for the cases $N=2$, $N=3$, and $N>3$ explicitly,

$$S_n^{(N=2)} = -\beta\mathcal{N}_\perp\frac{1+n}{n}+\frac{\beta^2\mathcal{N}_\perp}{2N^2}\left(\frac{1+n+n^2}{n^2}\right)+O(\beta^4), \tag{20}$$

$$S_n^{(N=3)} = -\beta\mathcal{N}_\perp \frac{1+n}{n} + \frac{\beta^2\mathcal{N}_\perp}{4N^2}\left(\frac{1+n+n^2}{n^2}\right) + \frac{\beta^3\mathcal{N}_\perp}{24N^3}\frac{(1+n)(1+n^2)}{n^3} + O(\beta^4), \tag{21}$$

$$S_n^{(N>3)} = -\beta\mathcal{N}_\perp \frac{1+n}{n} + \frac{\beta^2\mathcal{N}_\perp}{4N^2}\left(\frac{1+n+n^2}{n^2}\right) + O(\beta^4). \tag{22}$$

As a result, the entanglement entropy of SU(N) gauge theory in the strong coupling expansion is

$$S_{\text{EE}}^{(N=2)} = \frac{A_\perp}{a^2}\left[-\frac{4N^2}{\lambda} + 6\frac{N^2}{\lambda^2} + O(\frac{N^4}{\lambda^4})\right] + \delta S_{\text{EE}}^{(N=2)}, \tag{23}$$

$$S_{\text{EE}}^{(N=3)} = \frac{A_\perp}{a^2}\left[-\frac{4N^2}{\lambda} + 3\frac{N^2}{\lambda^2} + \frac{4}{3}\frac{N^3}{\lambda^3} + O(\frac{N^4}{\lambda^4})\right] + \delta S_{\text{EE}}^{(N=3)}, \tag{24}$$

$$S_{\text{EE}}^{(N>3)} = \frac{A_\perp}{a^2}\left[-\frac{4N^2}{\lambda} + 3\frac{N^2}{\lambda^2} + O(\frac{N^4}{\lambda^4})\right] + \delta S_{\text{EE}}^{(N>3)}, \tag{25}$$

where $A_\perp$ is the area of A, B region interface, a is lattice spacing, and $\lambda = g^2 N$ is the t'Hooft coupling. As we argued below Eq. (11), this result is independent of how the lattice is discretized, as long as the conical singularity is encircled by the same number of plaquettes in the action. The additional terms δS_{EE} are the positive contribution from the cosmological constant living on the 2-dimensional space transverse to the conical singularity in the Lagrangian, which we expect to cancel the negative term in the entanglement entropy, and will be explained in Sec. 4.

4. A 2-d cosmological constant counterterm

In the result Eqs. (23)~(25), one finds that the leading order $O(\beta)$ term in the entanglement entropy has a negative contribution while all the subleading orders are positive. It turns out that, with the 2 dimensional conical structure in a 4 dimensional space time, we are allowed to introduce more local operators in the continuum action

$$S = \int d^2x_\perp d^2x_\parallel \left[-\frac{1}{4}F^2 + c_4 + c_2\delta^{(2)}(x_\parallel, \tau)\right]. \tag{26}$$

The c_4 term is a 4-dimensional cosmological constant counter term which does not contribute to the entanglement entropy. However, the 2-dimensional cosmological constant counter term c_2, living on the space transverse to the conical singularity and breaks the translational symmetry on the cone, can contribute to the entanglement entropy [24]. Assuming c_2 is a smooth function of n, then

$$c_2 = c_2'(n-1) + O\left((n-1)^2\right), \tag{27}$$

where we $c_2' = c_2'(\beta)$ is a function of β. We have made use of the fact that c_2 should vanish at $n = 1$ where translational symmetry is recovered. Therefore there is an extra unspecified contribution to the entanglement entropy which also obey the area law:

$$\delta S_{\text{EE}}^{(N)} = A_\perp c_2'^{(N)}. \tag{28}$$

We label the N dependence explicitly since different theories would have different c_2' counter terms. Also, $c_2'^{(N)}$ is β dependent.

The negative leading term in our entanglement entropy could in principle be compensated by the contribution from the 2-dimensional cosmological constant in (28). We remind the readers here that the negative leading term arises from the constant term in the lattice gauge field Lagrangian. Since the constant in the Lagragian for the central plaquettes is different from that for the non-central ones by a factor of $1/n$, the effect is just like the c_2 in Eq. (26). If these constant terms are not included in the lattice Lagrangian, like what was done in some of the actions studied previously [10,19,21], then the negative contribution to the entanglement entropy will not arise, and the Lagrangian can not be reduced to the usual Yang-Mills one at continuum limit. We argue that the c_2 and c_4 terms will always appear by renormalization even they are set to zero at certain renormalization scale. Setting them to be zero is equivalent to choosing specific values for these counter terms. Previously it was known that different choices of the boundary conditions gave different values for the entanglement entropy [15]. This corresponds to employing different regularization schemes, but those differences can be compensated by having different values for the counter terms for different regularization schemes used.

We further speculate that the 2-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Also, they could play the role of a counter term to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

In spite of the discussion above, we would like to remind the reader that, since there exist many possible center choices which may not necessarily lead to the constant counter terms at the tip of the cone, the inclusion of our proposed cosmological constants may not be sufficient to achieve full agreement with the results from various center choices.

5. Comparison with previous results

Calculating the entanglement entropy of the guage fields is a subtle problem because of the gauge invariance constraint and the nonlocality of gauge fields on the lattice. Various approaches ([11], [13–18]) had been proposed to tackle this problem, and [18] had pointed out that these approaches are equivalent from the gauge invariant local operator algebraic perspective [15]. [15] elaborate the choice of the electric, magnetic, and trivial centers in the operator algebra, corresponding to different choice of the boundary condition for the local Hilbert space. The setup of our system, as illustrated in Fig. 2, has the boundary cutting across the links connecting the regions A and B. This configuration is very similar to that in [16], except that ours has one more spatial dimension along the boundary plane. The construction in [16] had been demonstrated in [15] being equivalent to the electric center choice in the algebraic approach, as they give rise to the same expectation value to the same local operators. We remind the readers again that the non-vanishing entanglement entropy in our model arises from the existence of the central plaquette which encircles the conical singularity on the Riemannian n-sheets in our the configuration, and this is because we choose to place the boundary in between two adjacent lattice sites, allow it to cut through the links connecting two different regions.

Our result shows that the entanglement entropy between regions A and B of $SU(N)$ Yang-Mills gauge fields on the lattice obeys the area law up to the highest order $O(\beta^3)$. At zero temperature and large coupling, such system is expected to be confined. In the literature, however, the area law is well known for the weak coupling theories and conformal theories, and it is not yet well-established whether the area law extends to the strong coupling and confining regime. Our result in analytic formalism under the strong coupling expansion suggests that the area law applies to the confining strong coupling theories. This echoes [18] which demonstrates the entanglement entropy of the strongly coupled $SU(N)$ lattice gauge theory behaves as $|\partial V_\perp|\frac{\log g^2 N}{g^8 N^2}$ at large N. On the other hand, [26–28] compute the C-function $C(l) = \frac{l^3}{|\partial A|}\frac{\partial S_{EE}}{\partial l}$ of the entanglement entropy for the Yang-Mills theory by lattice simulation in $3+1$ dimensions, where l is the size of the domain A they calculate the S_{EE} for. The results show that for sufficiently large l, $C(l)$ vanishes. Since $C(l)$ is normalized by the boundary area $|\partial A|$ of the domain A, the results imply that the entanglement entropy scales with the boundary area. The evidences from lattice simulation also suggestion that the area law applies to the confining Yang-Mills theories on the lattice, in accordance with our result.

Our system is of infinite size in space and time dimensions. In [19], the entanglement entropy of SU(N) gauge fields on the lattice of finite size at finite temperature is calculated in the Lagrangian formalism by means of group characteristic expansions. In $1+1$ dimensions, when the periodic boundary condition is imposed on the spatial dimension, the entanglement entropy is given by

$$S_{\text{ent.}} = \left(\frac{\beta}{2N^2}\right)^{A/a^2}\left[1 - \log\left(\left(\frac{\beta}{2N^2}\right)^{A/a^2}/N^2\right)\right] \tag{29}$$

where A is the total area of the $1+1$ dimensional spacetime and a is the lattice spacing. At the limit of infinite area, the entanglement entropy reduces to 0, instead of Eqs. (23)~(25). While the free boundary condition is chosen in the spatial dimensions, the entanglement entropy vanishes identically. We can obtain the same result if we set the conical singularity on a site so there is no central plaquette at all. So the leading order result for a system of periodic boundary condition in $1+1$ dimensions comes from the configuration with the plaquettes tiling the whole dimensions. But this configuration gives vanishing entanglement entropy when the free spatial boundary condition is taken. Despite this, the result in Eq. (29) does not have the area law due to the nature of 1 spatial dimensional system, and hence does not have the expected form.

Eqs. (23)~(25) show the leading UV-divergence $\sim \frac{A_\perp}{a^2}$ of the Yang-Mills entanglement entropy in our setup, where the lattice spacing a is the cutoff of the system. It agrees with the leading divergence of S_{EE} for the conformal theories in $3+1$ dimensions. It would be interesting to investigate whether the entanglement entropy of the Yang-Mills fields (or even of other gauge theories, e.g. Z_2 gauge fields) also exhibits logarithmic scaling in the subleading universal terms in our setup, as our future work.

According to our result in Eq. (25), the strongly coupled $SU(N)$ Yang-Mills gauge theory at the large-N limit in our setup behave as $S_{\text{EE}}^{(N>>1)} = -\frac{A_\perp}{a^2}\frac{4N^2}{\lambda} = -\frac{A_\perp}{a^2}\left[\frac{4N}{g^2}\right]$. In Section 4, we explained that this term could be compensated by the contribution from the 2-dimensional cosmological constant in (28). The coefficient of this leading $S_{\text{EE}}^{(N>>1)}$ per unit entangling surface area vanishes if we allow N and g^2 to approach infinity in such a way that $\frac{4N}{g^2} \to 0$. We can compare our result to that in [18], where the S_{EE} of the strongly coupled $SU(N)$ lattice gauge theory scales as $|\partial V_\perp|\frac{\log(\lambda N)}{\lambda^2} \sim |\partial V_\perp|\frac{\log(g^2 N)}{g^8 N^2}$ at large N, derived from the Hamiltonian approach. Note that [18] sets lattice spacing to 1 and takes $\lambda \sim g^4 N$, while in our work we use $\lambda = g^2 N$. Since the replicating links and the central plaquette prescription are not derivable from the Hamiltonian formalism, it is not surprising that our aforementioned leading scaling behavior in S_{EE} does not agree with that in [18].

Ref. [14] considers the entanglement entropy of the ground states of the SU(2) Kogut-Susskind Hamiltonian [29] for the Wilson gauge theories at strong coupling limit, by including the edge states living on the boundary into the Hilbert space, such that the total entanglement entropy contains the contribution from the edge states. For $d + 1 \geq 3$ dimensions, the leading order entanglement entropy obtained by [14] is

$$S_{\text{ent.}} = \frac{A_\perp}{a^2}(d-1)\beta^2(\ln\frac{1}{\beta^2} + 1 + 2\ln 2) \tag{30}$$

where $A_\perp$ is the boundary area.[2] The entire leading contribution to the entropy in (30) is given by the entanglement entropy of the edge modes, which corresponds to the contribution of the cosmological constant counterterm in our model. The non-local correlations of the d.o.f.'s in the two subregions are manifest only at higher order. Ref. [16] also gives similar result: it is demonstrated via numerical simulation that, in the case of the Z_2 lattice gauge theory in 3 dimensions, the entanglement entropy is almost saturated by the entropy of the end points of the electric strings cut open by the boundary of the two subregions. In our language, their setup corresponds to locating the conical singularity on a site, so there is no central plaquette at all such that the leading contribution is coming from the counterterm.

In our case, introducing the central plaquette incorporates the effect of the edge modes on the boundary. In principle it is possible to extract the dynamics of electric flux and magnetic flux on the boundary, and the remaining contribution of the edge modes is naturally interpreted as the cosmological constant counter-terms in the theory. This is left for our future work.

6. Summary

To summarize, we have calculated the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the n-sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term is negative, which could in principle be canceled by taking into account the cosmological constant living in interface of the two entangled subregions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We have further speculated that this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[2] Note that the β^2 in [14] is equivalent to our β. Eq. (30) is expressed in our notation.

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