Cosmology with an Adapted System of Coordinates

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Most of the observables in cosmology are light-propagated quantities. I will present a system of coordinates, the "geodesic light-cone (GLC) coordinates", which is adapted to the propagation of null-like signals observed by a geodesic observer. This system is fully general and adapted to calculations in inhomogeneous geometries. I will show that its properties make it useful for a large spectrum of applications, from evaluating the distance-redshift relation, calculating averages on our past light cone, estimating the effect of the large-scale structure on the Hubble diagram, to calculating weak lensing quantities.

Keywords: Inhomogeneous cosmology; general relativity; gravitational lensing.

1. Motivations

Except for planet Earth from which we have geological information, the vast majority of observations available to us about our Universe comes from our past light cone. It is thus natural to use coordinates adapted to it. We present coordinates which were first developed to simplify averages of scalars on our past light-cone,[1] next used to estimate the effect of inhomogeneities on luminosity distance[2–5] and Hubble diagram,[6,7] and recently applied to compute lensing quantities.[8]

2. The geodesic light-cone coordinates

Our light-cone adapted metric consists of 6 arbitrary functions (Υ, U^a, γ_{ab}) and is close to "*observational coordinates*"[9,10] (but different[11]) and totally gauged fixed:

$$\mathrm{d}s^2_{\mathrm{GLC}} = \Upsilon^2 \mathrm{d}w^2 - 2\Upsilon \mathrm{d}w \mathrm{d}\tau + \gamma_{ab}(\mathrm{d}\widetilde{\theta}^a - U^a \mathrm{d}w)(\mathrm{d}\widetilde{\theta}^b - U^b \mathrm{d}w) \quad . \tag{1}$$

This metric uses a time coordinate τ corresponding to the proper time of an observer in geodesic motion, a null coordinate w describing its past light cones, and finally angles $\tilde{\theta}^a$ $(a = 1, 2)$ that photons keep along their path orthogonal to a 2-spheres $\Sigma(w, \tau)$ of constant time in our past light-cone (see Fig. 1). A possible simple choice in the FLRW limit is: $w = \eta + r$ (conformal time + radius), $\tau = t$ (cosmic time), $(\tilde{\theta}^1, \tilde{\theta}^2) = (\theta, \phi)$, $\Upsilon = a(t)$, $U^a = 0$, $\gamma_{ab} = a^2 r^2 \mathrm{diag}(1, \sin^2\theta)$. Υ can be seen as an inhomogeneous scale factor (lapse function), U^a is similar to a shift-vector and γ_{ab} is the metric inside $\Sigma(w, \tau)$.[11] The computation of the distance-redshift relation in GLC coordinates is facilitated by two direct simplifications:

$$\text{Redshift:} \quad (1 + z_s) = \Upsilon(w_o, \tau_o, \widetilde{\theta}^a)/\Upsilon(w_o, \tau_s, \widetilde{\theta}^a) \quad , \tag{2}$$

$$\text{Angular distance:} \quad d_A = \gamma^{1/4}(\sin\tilde{\theta}^1)^{-1/2} \quad \text{with } \gamma \equiv \det(\gamma_{ab}) = \frac{|\det(g_{\mathrm{GLC}})|}{\Upsilon^2} \ . \tag{3}$$

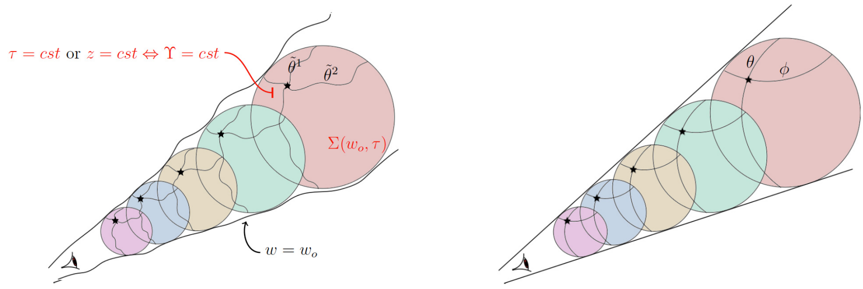


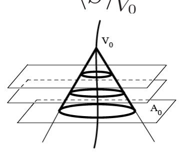
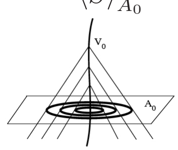
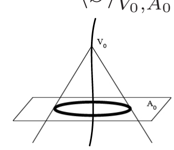
Fig. 1. *Left:* Inhomogeneous light-cone parametrized by GLC coordinates. *Right:* Homogeneous light-cone or an adapted system (like GLC coordinates) in an inhomogeneous geometry.

3. Simplification of light-cone averages

The general expression of the light-cone average[1] of a scalar S (like d_L, d_L^{-2}) is given by $\langle S \rangle (V_0, A_0) = I(S; V_0, A_0)/I(1; V_0, A_0)$. We define the average integral such that it is gauge invariant, invariant under reparametrisations of the form $(A, V) \to (\tilde{A}(A), \tilde{V}(V))$ and under general coordinate transformations $x \to y$:

$$I(S; V_0, A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g}\; D(V_0 - V) D(A - A_0)\; \mathcal{N}(V, A, \partial_\mu)\; S(x) \quad , \qquad (4)$$

where $\mathcal{N}(V, A, \partial_\mu)$ is a normalization, $D(X) = \delta_D(X)$ or $\Theta(X)$ (Heaviside function). From that definition, three configurations are possible (null, spatial, spherical):

Average $^{\Theta}_{\delta_D}$	$\langle S \rangle^{A_0}_{V_0}$	$\langle S \rangle^{V_0}_{A_0}$	$\langle S \rangle_{V_0, A_0}$
Illustration			
$\mathcal{N}(V, A, \partial_\mu)$	$\frac{\lvert\partial_\mu V \partial^\mu A\rvert}{\sqrt{-\partial_\nu A \partial^\nu A}}$	$\sqrt{-\partial_\mu A \partial^\mu A}$	$\lvert\partial_\mu V \partial^\mu A\rvert$

Among these three types of averages, $\langle S \rangle_{V_0, A_0}$ is closer to physical observables as it averages over the deformed 2-sphere embedded in the light-cone $V = V_0$ and a spatial hypersurface $A = A_0$. In GLC coordinates (where $V \to w$, $A \to \tau$) we can simplify the average and use Eq. (2) to get (with $\tau_z \equiv \tau(z_s, w_o, \widetilde{\theta}^a)$):

$$\langle S \rangle_{w_o, z_s} = \left(\int d^2\widetilde{\theta} \sqrt{\gamma(w_o, \tau_z, \widetilde{\theta}^b)}\, S(w_o, \tau_z, \widetilde{\theta}^b) \right) \Big/ \left(\int d^2\widetilde{\theta} \sqrt{\gamma(w_o, \tau_z, \widetilde{\theta}^b)} \right) \quad , \qquad (5)$$

allowing us to average scalars on the observed sky, at a certain redshift z_s.

4. Distance-redshift relation at $\mathcal{O}(2)$

The GLC metric enables the computation of the luminosity distance $d_L(z)$ at $\mathcal{O}(2)$ in perturbations within the well-known Newtonian gauge (NG):

$$ds^2_{\rm NG} = a^2(\eta) \left[-(1 + 2\Phi) d\eta^2 + (1 - 2\Psi)(dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\phi^2]) \right] \qquad (6)$$

with $\Phi = \psi + \frac{1}{2}\phi^{(2)}$, $\Psi = \psi + \frac{1}{2}\psi^{(2)}$ the gauge invariant Bardeen potentials (assuming no matter shear at $\mathcal{O}(1)$). Doing the full $\mathcal{O}(2)$ transformation between GLC and NG coordinates brings the relations $(\tau, w, \widetilde{\theta}^1, \widetilde{\theta}^2) = \text{func}(\eta, r, \theta, \phi)$ and $(\Upsilon, U^a, \gamma^{ab}) = \text{func}(\psi, \psi^{(2)}, \phi^{(2)})$. These last allow to compute $d_L = (1+z)^2 d_A$, using Eq. (3). The first order is given by different terms containing ψ_s (at the source), (Integrated) Sachs-Wolfe ([I]SW), Doppler, and lensing (convergence) effects. Similarly obtained, the $\mathcal{O}(2)$-corrections contain[2, 6, 11–13]: two dominant terms (Doppler)^2, (Lensing)^2; combinations of $\mathcal{O}(1)$-terms like ψ_s^2, ([I]SW)^2, [I]SW $\times$ Doppler; genuine $\mathcal{O}(2)$-terms such as $\psi_s^{(2)}$, Lensing$^{(2)}$; new integrated effects, angle deformations; redshift perturbations from Eq. (2) (including transverse peculiar velocity); and terms like Lens-Lens coupling and corrections to the Born approximation.

5. Effect of large-scale structure on the Hubble diagram

In Sec. 4 the distance d_L was expressed in terms of $(\psi, \psi^{(2)}, \phi^{(2)})$, hence it needs a description of the Bardeen potentials at $\mathcal{O}(1,2)$. We expand the first order gravitational potential ψ in Fourier modes and denote by $\overline{(...)}$ the ensemble (or stochastic) average over perturbations. The $\mathcal{O}(2)$ potentials can be related to ψ by[14]: $\psi^{(2)}, \phi^{(2)} \propto \nabla^{-2}(\partial_i\psi\partial^i\psi), \partial_i\psi\partial^i\psi$. Hence the combination of light-cone and stochastic averages enables us to study the effect of matter perturbations on the whole sky (and sky-averaged observables like d_L). For example, the (trivial) average of ψ^2 gives: $\overline{\langle\psi_s^2\rangle} = \int_0^\infty \frac{dk}{k}\mathcal{P}_\psi(k)$ where $\mathcal{P}_\psi(k) \equiv (k^3/2\pi^2)|\psi_k(\eta)|^2 = (3/5)^2 A(k/k_0)^{n_s-1} T^2(k)\, g^2(z)$ is the power spectrum describing perturbations. At linear order, with A, n_s, k_0 taken from WMAP, $T(k)$ is a transfer function[15] including a baryonic component (Silk damping), and $g(z)$ is the growth factor describing the recent time evolution of perturbations.[11] In CDM we get exactly the spectral coefficients coming from each correction of $d_L(z_s, \theta^a)$ described in Sec. 4: $\overline{\langle d_L\rangle} = \int_0^\infty \frac{dk}{k}\mathcal{P}_\psi(k)\, C(k\Delta\eta)$. We do the same in ΛCDM, with reasonable assumptions to simplify integrations,[7] using also a non-linear power spectrum.[16, 17]

Like d_L one can also average the flux $\Phi = L/(4\pi d_L^2) \simeq \Phi_0 + \Phi_1 + \Phi_2$. We get:

- $\overline{\langle d_L^{-2}\rangle} \equiv (d_L^{FLRW})^{-2}\,[1 + f_\Phi(z)]$ where $f_\Phi(z) \simeq f(z)\int_0^\infty \frac{dk}{k}\left(\frac{k}{\mathcal{H}_0}\right)^2 \mathcal{P}_\psi(k)$,
- $\overline{\langle d_L\rangle}(z) = d_L^{FLRW}\,[1 + f_d(z)]$ with $f_d = -(1/2)f_\Phi + (3/8)\overline{\langle(\Phi_1/\Phi_0)^2\rangle}$.

The d_L-correction involves the flux variance lead by peculiar velocity and lensing. Similarly, we get the average/dispersion of the distance modulus $\mu = 5\log_{10} d_L$:

$$\overline{\langle\mu\rangle} = \mu^{\text{FLRW}} - 1.25(\log_{10} e)\left[2f_\Phi - \overline{\langle(\Phi_1/\Phi_0)^2\rangle}\right] , \tag{7}$$

$$\sigma_\mu = 2.5(\log_{10} e)\sqrt{\overline{\langle(\Phi_1/\Phi_0)^2\rangle}} . \tag{8}$$

Compared to the Union 2 data and using a non-linear power spectrum in ΛCDM (Fig. 2, Left), we find that peculiar velocities explain well the scatter at small z and

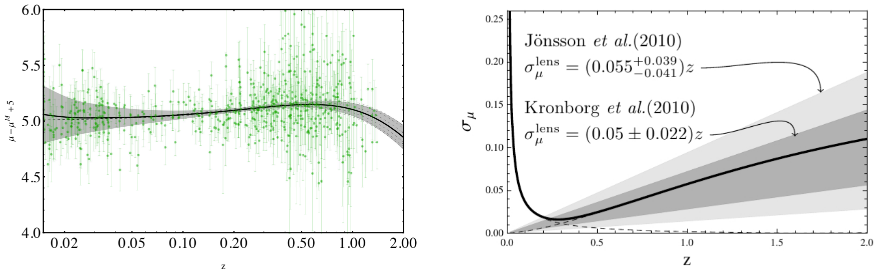


Fig. 2. *Left:* Theoretical dispersion from power spectrum vs data. *Right:* Dispersion σ_μ from peculiar velocities and lensing (solid line) compared to experimental estimates (grey areas).

that lensing only partially explains the scatter at large z. Finally, we can compare our dispersion on the Hubble diagram with the experimental estimations coming from lensing (Fig. 2, Right). We find that the total effect is well fitted by Doppler ($z \leq 0.2$) + lensing ($z > 0.3$) effects and that the lensing prediction, in great agreement with experiments so far,[18,19] should be confirmed by future surveys.

6. Jacobi map and weak lensing

We now consider lensing,[8] motivated by Sec. 5 and recent work[20] on galaxy number counts in GLC. The separation of two light rays emitted at the same time from a source S and converging to an observer O follows the geodesic deviation eq. (GDE):

$$\nabla^2_\lambda \xi^\mu = R^\mu_{\alpha\beta\nu} k^\alpha k^\nu \xi^\beta \tag{9}$$

with $\nabla_\lambda \equiv \mathrm{D}/\mathrm{d}\lambda \equiv k^\mu \nabla_\mu$, k^μ the photon momentum, λ an affine parameter along the photon path, ξ^μ the orthogonal displacement of the rays. The GDE is projected on the Sachs basis $\{s^\mu_A\}_{A=1,2}$ (two zweibeins with flat index $A = 1, 2$): $g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB}, s^\mu_A u_\mu = 0, s^\mu_A k_\mu = 0, \Pi^\mu_\nu \nabla_\lambda s^\nu_A = 0$; with u_μ the peculiar velocity of the comoving fluid (S, O comoving too), Π^μ_ν a "screen" projector orthogonal to u_μ and $u_\mu + (u^\alpha k_\alpha)^{-1} k_\mu$. We define the Jacobi map J^A_B, from the observed sky angle $\bar\theta^A_o$ to $\xi^A = \xi^\mu s^A_\mu$, by $\xi^A(\lambda) = J^A_B(\lambda, \lambda_o)\, \bar\theta^A_o$. The projected quantities ξ^A and $R^A_B = R_{\alpha\beta\nu\mu} k^\alpha k^\nu s^\beta_B s^\mu_A$ (optical tidal matrix) bring us the Jacobi equation[21,22]:

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} J^A_B(\lambda, \lambda_o) = R^A_C(\lambda)\, J^C_B(\lambda, \lambda_o) \quad , \tag{10}$$

$$\text{with I.C.:} \quad J^A_B(\lambda_o, \lambda_o) = 0 \quad , \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} J^A_B(\lambda_o, \lambda_o) = (k^\mu u_\mu)_o\, \delta^A_B \quad . \tag{11}$$

A direct resolution of Eq. (10) gives the angular distance of the source $d_A(\lambda_s) \equiv \sqrt{dS_s/d^2\Omega_o} = \sqrt{\det J^A_B(\lambda_s, \lambda_o)}$. Also, the (unlensed) angular position of the source $\bar\theta^A_s$ and the observed lensed position $\bar\theta^A_o$ (of the image) are given by: $\bar\theta^A_s = \left(\xi^A/\bar d_A\right)_s$, $\bar\theta^A_o = \left(k^\mu \partial_\mu \xi^A / k^\mu u_\mu\right)_o$, where $\bar d_A$ is the homogeneous and isotropic background our model refers to. This allows us to define the so-called amplification matrix as:

$$\mathcal{A}^A_B \equiv \frac{\mathrm{d}\bar\theta^A_s}{\mathrm{d}\bar\theta^B_o} = \frac{J^A_B(\lambda_s, \lambda_o)}{\bar d_A(\lambda_s)} = \begin{pmatrix} 1 - \kappa - \hat\gamma_1 & -\hat\gamma_2 + \hat\omega \\ -\hat\gamma_2 - \hat\omega & 1 - \kappa + \hat\gamma_1 \end{pmatrix} \quad , \tag{12}$$

which defines the lensing quantities: κ (convergence), $\hat{\omega}$ (vorticity), $|\hat{\gamma}| \equiv \sqrt{(\hat{\gamma}_1)^2+(\hat{\gamma}_2)^2} = \sqrt{(1-\kappa)^2+\hat{\omega}^2-\mu^{-1}}$ (shear), $\mu \equiv 1/(\det \mathcal{A})$ (magnification).

Let us now turn to the GLC coordinates and express these lensing quantities in it. First, the zweibeins are written as $s^\mu_A = (s^\tau_A, 0, s^a_A)$ and $k^\mu \equiv \omega \Upsilon^{-1} \delta^\mu_\tau$ (where ω is a pure constant). Second, the solution to Eqs. (10) and (11) in GLC is:

$$J^A_B(\lambda,\lambda_o) = s^A_a(\lambda)\ \left[2u_\tau (\dot{\gamma}_{ab})^{-1}\right]_o s^B_b(\lambda_o) \tag{13}$$

where $(\ldots)^{\bullet} \equiv \partial_\tau(\ldots)$. The angular distance and the magnification become:

$$d_A = 2u_{\tau_o}(\gamma\gamma_o)^{1/4}/\sqrt{(\det\dot{\gamma}_{ab})_o} \quad , \quad \mu = \left(\bar{d}_A/d_A\right)^2 = \Phi/\bar{\Phi} \ , \tag{14}$$

involving $\bar{d}_A = a^2(\tau)r^2$ with $r = w - \int \mathrm{d}\tau/a(\tau)$ measured from the observer and Φ ($\bar{\Phi}$) the flux in the inhomogeneous (homogeneous) geometry. The expression of the zweibeins can also be obtained in the GLC coordinates,[8] but it is more convenient to compute the squared lensing quantities, combined with $s^A_a s^A_b = \gamma_{ab}$ and $\epsilon_{AB}\, s^A_a s^B_b = \sqrt{\gamma}\,\epsilon_{ab}$ (ϵ the anti-symmetric symbol), to directly get:

$$\begin{pmatrix} (1-\kappa)^2+\hat{\omega}^2 \\ \hat{\gamma}_1^2+\hat{\gamma}_2^2 \end{pmatrix} = \left(\frac{u_{\tau_o}}{\bar{d}_A}\right)^2 \left(\left[\frac{\gamma\,\dot{\gamma}_{ab}\gamma^{bc}\dot{\gamma}_{cd}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)^2}\right]_o \gamma\,\gamma^{ad} \pm \frac{2\sqrt{\gamma\,\gamma_o}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)_o} \right) \ . \tag{15}$$

We thus have general lensing quantities expressed with only 3 metric functions (of γ_{ab}), showing the great advantage of using GLC coordinates. Similar expressions for the deformation matrix $\mathcal{S}^A_B \equiv \frac{\mathrm{d}J^A_C}{\mathrm{d}\lambda}(J^{-1})^C_B$ and its elements (optical scalars) simplify in GLC (depending only on γ_{ab}, Υ and their time derivatives). The usefulness of GLC coordinates to compute lensing quantities was illustrated[8] for an off-center observer in a Lemaître-Tolman-Bondi model (considering only the decaying mode).

7. Conclusions

We have briefly described the many advantages in using the GLC coordinates. They are indeed adapted to calculations involving light-propagation. They can also be used for weak lensing (where γ_{ab} acts as a screen), and help to get new predictions on cosmology[23] or study other aspects of weak (and possibly strong) lensing.

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