New definition of wormhole throat

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We present a new definition of the wormhole throat including the flare-out condition and the traversability for general dynamical spacetimes in terms of null geodesic congruences. We will examine our definition for some examples and see advantages compared to the others.

I. INTRODUCTION

Wormhole is one of the interesting objects in general relativity [1–4]. Although we have a vague image of wormhole, there is no universal definition which can work for general situations. To discuss wormhole we have to specify the throat, the flare-out condition and so on. There are some proposals [1, 5–7]. As far as we know, for static and spherical symmetric cases, the definition of the throat was firstly given in Ref. [1]. Therein, the static slices are embedded to the Euclid space to see the throat structure. Since the symmetries are used for the definition of wormhole, it is clear that this proposal is not applicable for dynamical or non-spherical symmetric cases.

The issue about the extension of the concept of wormhole to general spacetimes has been already addressed in Refs. [5, 6] (see also Refs. [7–11]). In Refs. [5, 6], using null geodesic congruences, the wormhole throat is defined as the minimal surface on the null hypersurfaces, i.e. the trapping horizon [12]. We know that, in their definition, some exotic matters are required for static wormholes [1, 2, 8, 11, 13] and for dynamical ones [5, 9, 10] without any singularities. Nevertheless, our Universe has the initial singularity, and cosmological wormhole solutions with the initial singularity, where two Friedmann-Lemaître-Robertson-Walker (FLRW) universes are connected, were constructed without any exotic matters [7]. Since these solutions do not meet the definition of Refs. [5, 6], the authors proposed an alternative definition focusing on spherical symmetric cases; the wormhole throat is the minimal surface on spacelike hypersurfaces [7]. However, this definition strongly depends on which spacelike hypersurfaces we take and, because of this dependence, even de Sitter and FLRW spacetimes are categorized into wormhole.

In this paper, we would propose a new definition of the wormhole throat which is better suited for the intuitive image of wormhole. Our definition seems to be a hybrid one of null-hypersurface-based definition [5, 6] and spacelike hypersurface one [7]. We describe the throat in terms of the expansion rate of null geodesic congruences on a kind of spacelike hypersurface.

The other parts of this paper are organized as follows. In Sec. II, we introduce a new definition of the wormhole throat and discuss some general features. In Sec. III, we consider several examples to see if our definition can work well. Finally, we give a summary in Sec. IV.

II. NEW DEFINITION

In this section, we propose a new definition of the wormhole throat with the flare-out condition and the traversability. We also discuss some general features.

We consider a codimension two spacelike compact surface S and future directed outgoing/ingoing null geodesic congruences with the affine parameter $\lambda_\pm$ emanating from S. Then we define the null expansion rate $\theta_\pm$ and we introduce the following quantities

$$k := \theta_+ - \theta_- \tag{1}$$

and

$$\bar{k} := \theta_+ + \theta_-. \tag{2}$$

Defining the following two vectors

$$r^a := (\partial_+ - \partial_-)^a \tag{3}$$

and

$$t^a := (\partial_+ + \partial_-)^a, \tag{4}$$

where $\partial_\pm := \partial_{\lambda_\pm}$, k and $\bar{k}$ are rewritten as

$$k = r^a \nabla_a \ln \sqrt{h}, \quad \bar{k} = t^a \nabla_a \ln \sqrt{h}. \tag{5}$$

In the above h is the determinant of the induced metric of the codimension two surface S. Although we cannot assume that the affine parameter $\lambda_\pm$ emanating from S provides us the global coordinate for spacetimes in general, we can have a quasi-local null coordinate system $\tilde{\lambda}_\pm$ such that it coincides with $\lambda_\pm$ when it crosses S.

Now we define the throat as the codimension two surface such that

$$k|_S = 0 \tag{6}$$

holds and the following flare-out condition

$$r^a \nabla_a k|_S > 0 \tag{7}$$

is satisfied. We emphasize that, by fixing the coordinate locally with $\tilde{\lambda}_\pm$, there is no ambiguity of the spatial derivative $r^a\nabla_a$. To introduce the concept of the traversability for the wormhole we consider the time sequence of the throat. We say that the wormhole is traversable if the tangent vector of the sequence of the throat, which is normal to S, is timelike. Moreover, if there are some event horizons in the region that satisfies Eq. (6) and inequality (7), we exclude the inside of the event horizon from the definition of the throat. This is because travelers cannot come back to the same region from black hole.

Let us look at general properties of our definition of the wormhole. From the condition of Eq. (6) we have

$$\theta_+|_S = \theta_-|_S. \tag{8}$$

When $\theta_+|_S = \theta_-|_S < 0$ (> 0), it means the existence of the future(past) trapped surface. Then, if the null energy condition holds, the singularity theorem implies the presence of singularity in the future(past) [14]. With the energy condition, assuming the cosmic censorship conjecture to be held, the future trapped region is always inside the event horizon [14], and thus it is excluded in our definition of wormhole. Meanwhile, with $\theta_+|_S = \theta_-|_S > 0$, the singularity theorem predicts the past singularity but we do not exclude the past trapped region. The realisation of $\theta_-|_S > 0$ will be easy in expanding universe and the past singularity may be unified to the initial one. That is, there is a room to construct a dynamical wormhole in the cosmological context keeping the energy condition.

Let z^a to be the tangent vector of the time sequence of the throat which is normal to S. Since it is timelike, we can write it as $z^a = \alpha(\partial_+)^a + \beta(\partial_-)^a$ with $\alpha, \beta > 0$. Along the time sequence,

$$z^a\nabla_a k|_S = 0 \tag{9}$$

holds and this gives

$$\partial_- k|_S = -\frac{\alpha}{\beta}\partial_+ k|_S. \tag{10}$$

Then, $r^a\nabla_a k|_S$ becomes

$$\begin{aligned}
r^a\nabla_a k|_S &= \left(1 + \frac{\alpha}{\beta}\right)\partial_+ k|_S \\
&= \left(1 + \frac{\alpha}{\beta}\right)(\partial_+\theta_+ - \partial_+\theta_-)|_S > 0. \tag{11}
\end{aligned}$$

If the null energy condition is satisfied in D-dimensional spacetimes, the Raychaudhuri equation tells us

$$\partial_+\theta_+ = -\frac{1}{D-2}\theta_+^2 - \sigma_{+ab}\sigma_+^{ab} - R_{ab}n^a n^b \leq 0, \tag{12}$$

where σ_{+ab} is the shear and n^a is the tangent vector of null geodesics. Here we used the fact that the null geodesic congruences are normal to the throat, that is, the rotation of the congruence vanishes. Therefore,

$$\partial_+\theta_-|_S < 0 \tag{13}$$

is required at least for the presence of the throat.

It is nice to have other general features. Since we have an equality

$$r^a\nabla_a k + t^a\nabla_a \bar{k} = 2(\partial_+\theta_+ + \partial_-\theta_-), \tag{14}$$

the Raychaudhuri equation with the null energy condition show us

$$r^a\nabla_a k + t^a\nabla_a \bar{k} \leq 0. \tag{15}$$

In particular, the flare-out condition is not satisfied, $r^a\nabla_a k \leq 0$, for a sort of static case of $t^a\nabla_a \bar{k} = 0$ as long as the null energy condition is satisfied. This is a simple confirmation of well-known fact.

III. EXAMPLES

Let us examine our definition in four dimensional spacetimes with symmetries including the spherical symmetry.

A. General scheme

In the null coordinate, the metric of a spherically symmetric spacetime is generically written as

$$ds^2 = -a^2(u,v)du\,dv + R^2(u,v)d\Omega_2^2, \tag{16}$$

where $d\Omega_2^2$ is the metric of the unit 2-sphere. The throat is supposed to be located at a two surface specified by $u = u_0, v = v_0$.

The radial null geodesic will be on u or v =constant lines. Let us consider the geodesic on $v = v_0$ which follows the geodesic equation

$$\frac{d^2u}{d\lambda_u^2} + 2\frac{\partial_u a}{a}\left(\frac{du}{d\lambda_u}\right)^2 = 0. \tag{17}$$

In a formal way, we can solve the above as

$$\lambda_u = C_u^{-1}\int^u a^2(u')du' =: U, \tag{18}$$

where $a(u) := a(u,v_0)$. C_u is the positive integration constant and we choose λ_u such that $du/d\lambda_u > 0$. In the same way, for the geodesic on $u = u_0$, we have

$$\lambda_v = C_v^{-1}\int^v a^2(v')dv' =: V, \tag{19}$$

where $a(v) := a(u_0,v)$. C_v is the positive integration constant and we choose λ_v such that $dv/d\lambda_v > 0$. Employing U, V as new coordinates, the metric (16) is rewritten as

$$ds^2 = -C_u C_v \frac{a^2(u,v)}{a^2(u)a^2(v)}dU\,dV + R^2(u,v)d\Omega_2^2. \tag{20}$$

We would stress that U (V) is the affine parameter on v_0 (u_0) =constant geodesic.

The null expansion rate θ_U, θ_V are calculated to be

$$\theta_U = \theta_- = \frac{2}{R}\partial_U R, \quad \theta_V = \theta_+ = \frac{2}{R}\partial_V R. \quad (21)$$

So k defined by Eq. (1) becomes

$$
\begin{aligned}
k &= \frac{2}{R}(\partial_V - \partial_U)R \\
&= \frac{2}{R}(C_v a^{-2}(v)\partial_v - C_u a^{-2}(u)\partial_u)R. \quad (22)
\end{aligned}
$$

On the throat S, k is supposed to vanish and then we have

$$C_u \partial_u R|_S = C_v \partial_v R|_S. \quad (23)$$

In the above, we used the fact of $a(u_0) = a(v_0) =: a_0$.

We also need to check the flare-out condition (7) and the traversability (9), which are written with the coordinate (20) as

$$
\begin{aligned}
&r^a \nabla_a k|_S \\
&= (\partial_V - \partial_U)k|_S \\
&= \frac{2}{a_0^4 R}\Big[-2C_v^2 \partial_v \ln a(v)\partial_v R - 2C_u^2 \partial_u \ln a(u)\partial_u R \\
&\qquad + C_v^2 \partial_v^2 R - 2C_u C_v \partial_u \partial_v R + C_u^2 \partial_u^2 R\Big]\Big|_S \\
&> 0 \quad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
&z^a \nabla_a k|_S \\
&= (\alpha\partial_V + \beta\partial_U)k|_S \\
&= \frac{2}{a^4 R}\Big[-2\alpha C_v^2 \partial_v \ln a(v)\partial_v R + 2\beta C_u^2 \partial_u \ln a(u)\partial_u R \\
&\qquad + \alpha C_v^2 \partial_v^2 R - (\alpha - \beta)C_u C_v \partial_u \partial_v R - \beta C_u^2 \partial_u^2 R\Big]\Big|_S \\
&= 0. \quad (25)
\end{aligned}
$$

Equalities (23), (25) and inequality (24) with the positivities of C_u, C_v, α and β are the conditions for the wormhole in spherically symmetric spacetimes.

B. Examples

In this subsection, we look at concrete examples which include non-wormhole spacetimes.

1. Schwarzschild spacetime

It is well-known that the throat of the Schwarzschild spacetime is not the throat of the wormhole due to the presence of the event horizon. Nevertheless, it is nice to see the feature in terms our definition of the throat. To see this we adopt the Kruskal coordinate

$$ds^2 = \frac{4r_g^3 e^{-r/r_g}}{r}(-dT^2 + dX^2) + r^2 d\Omega_2^2, \quad (26)$$

where $r_g = 2M$ and M is the Arnowitt-Deser-Misner(ADM) mass. The coordinate transformation from the Kruskal to ordinal one is given by

$$(r/r_g - 1)e^{r/r_g} = X^2 - T^2 \quad (27)$$

and

$$T/X = \tanh(t/2r_g) \quad (28)$$

for $r > r_g$, or

$$X/T = \tanh(t/2r_g) \quad (29)$$

for $0 < r < r_g$.

In this case, choosing u, v as $u = T - X, v = T + X$, $k|_S = 0$ (Eq. (23)) gives us

$$C_u(T + X)|_S = C_v(T - X)|_S. \quad (30)$$

This implies that the candidate of a throat is in the region $0 < r \leq r_g$ because of $C_u, C_v > 0$. In addition, inequality (24) and Eq. (25) become

$$r^a \nabla_a k|_S = \frac{4r_g^4}{a_0^4 r^4}e^{-r/r_g}C_u C_v|_S > 0, \quad (31)$$

$$z^a \nabla_a k|_S = \frac{2r_g^4}{a_0^4 r^4}e^{-r/r_g}(\alpha - \beta)C_u C_v|_S = 0, \quad (32)$$

where $a_0^2 = 4(r_g^3/r)e^{-r/r_g}$. The flare-out condition is satisfied as expected and the tangent vector of the throat orbit of $\alpha = \beta$ is timelike. However, there is the event horizon at $r = r_g$ that is the boundary of the region satisfying Eq. (23) and inequality (24). Therefore, as we have commented in Sec. II, the Schwarzschild spacetime does not have the wormhole throat.

2. de Sitter spacetime

Next we will examine the de Sitter spacetime. If one elaborates the selection of a spacelike hypersurface and follows Maeda et al.'s definition [7] for the wormhole throat, there is a case where the wormhole is. This is because Maeda et al.'s definition is not slightly appropriate. Meanwhile our definition excludes this case.

In the flat chart, the metric of the de Sitter spacetime is given by

$$
\begin{aligned}
ds^2 &= a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\Omega_2^2) \\
&= a^2(\eta)(-dudv + r^2 d\Omega_2^2), \quad (33)
\end{aligned}
$$

where $a(\eta) = -1/(H\eta)$, H is the Hubble constant and $u = \eta - r, v = \eta + r$. Then Eq. (23) implies

$$C_u(Har - 1)|_S = C_v(Har + 1)|_S. \qquad (34)$$

This has a solution

$$Har = \frac{C_u + C_v}{C_u - C_v} > 1, \qquad (35)$$

if one chooses C_u, C_v satisfying $C_u > C_v$. This means that the throat candidate is in outside of the cosmological horizon.

Let us see the flare-out condition (24). With the metric (33), we have

$$r^a \nabla_a k|_S = -\frac{2H^2}{a^2} C_u C_v|_S < 0. \qquad (36)$$

This disagrees with the flare-out condition (24). Therefore, there is no throat in the de Sitter spacetime as expected.

3. Friedmann-Lemaître-Robertson-Walker(FLRW) spacetime

Now we consider the FLRW spacetime. The metric is given by

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[(1 - kr^2)^{-1}dr^2 + r^2 d\Omega_2^2] \\ &= a^2(\eta)[-d\eta^2 + d\zeta^2 + r^2 d\Omega_2^2] \\ &= a^2(\eta)[-dudv + r^2 d\Omega_2^2], \end{aligned} \qquad (37)$$

where $k = -1, 0, 1$ depending on the spatial topology, η is the conformal time defined by $d\eta = a^{-1}(t)dt$, $d\zeta = dr/\sqrt{1 - kr^2}$ and $u = \eta - \zeta, v = \eta + \zeta$.

For the FLRW spacetime, Eq. (23) becomes

$$C_u(\dot{a}r - \sqrt{1 - kr^2})\Big|_S = C_v(\dot{a}r + \sqrt{1 - kr^2})\Big|_S, \qquad (38)$$

where $\dot{a} = da(t)/dt$. If one chooses C_u, C_v satisfying $C_u > C_v$, the above has the solution as

$$H(t)\frac{a(t)r}{\sqrt{1 - kr^2}} = \frac{C_u + C_v}{C_u - C_v} > 1, \qquad (39)$$

where $H(t) := \dot{a}(t)/a(t)$. Roughly speaking, as in the de Sitter case, this means that the throat candidate is in outside of the cosmological horizon.

For the current case, the flare-out condition (24) becomes

$$\begin{aligned} r^a \nabla_a k|_S &= \frac{2C_u C_v \left[a\ddot{a}(1 - kr^2) - \dot{a}^2 r^2(\dot{a}^2 + k)\right]}{a^4 \left[\dot{a}^2 r^2 - (1 - kr^2)\right]}\Big|_S \\ &> 0. \end{aligned} \qquad (40)$$

This requires

$$\dot{a}^2 r^2(\dot{a}^2 + k) < a\ddot{a}(1 - kr^2). \qquad (41)$$

Together with Eq. (39), the above implies

$$\dot{a}^2 r^2(\dot{a}^2 + k) < a\ddot{a}(1 - kr^2) < a\ddot{a}\dot{a}^2 r^2. \qquad (42)$$

Using the Friedmann equation, it is easy to see that the inequality $\dot{a}^2 + k < a\ddot{a}$ obtained from inequality (42) is equivalent with the violation of the null energy condition,

$$\rho + p < 0, \qquad (43)$$

where ρ and p are the energy density and the pressure of the perfect fluid, respectively. This is consistent with common sense.

4. Morris-Thorne wormhole

The Morris-Thorne wormhole, which is static and spherically symmetric, is often investigated [1–4, 11, 13]. The metric is given by

$$\begin{aligned} ds^2 &= -e^{2\Phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \\ &= e^{2\Phi}(-dt^2 + d\zeta^2) + r^2 d\Omega_2^2 \\ &= -e^{2\Phi}dudv + r^2 d\Omega_2^2, \end{aligned} \qquad (44)$$

where $\Phi(r), b(r)$ are functions of r, ζ is defined by $d\zeta = e^{-\Phi}dr/\sqrt{1 - \frac{b}{r}}$ and $u = t - \zeta, v = t + \zeta$. Here we suppose that $g_{tt} = -e^{2\Phi}$ is negative and regular. Note that this metric is not obtained as a solution of the Einstein equation with a given matter field action.

For this spacetime, Eq. (23) becomes

$$-C_u\sqrt{1 - \frac{b}{r}}\, e^\Phi\Big|_S = C_v\sqrt{1 - \frac{b}{r}}\, e^\Phi\Big|_S. \qquad (45)$$

This implies that the throat candidate is the surface that satisfies $b(r) = r$ because of $C_u, C_v > 0$. The flare-out condition (24) becomes

$$r^a \nabla_a k|_S = \frac{(C_u + C_v)^2(1 - b')}{4r^2}e^{-2\Phi}\Big|_S > 0, \qquad (46)$$

where $b' = db(r)/dr$. This flare-out condition is satisfied if

$$b' < 1 \qquad (47)$$

on S. The traversability (25) becomes

$$\begin{aligned} z^a \nabla_a k|_S &= \frac{(\alpha C_v - \beta C_u)(C_v + C_u)(1 - b')}{4r^2}e^{-2\Phi}\Big|_S \\ &= 0. \end{aligned} \qquad (48)$$

From this equation, we see that the tangent vector of the throat orbit of $\alpha C_v = \beta C_u$ is timelike.

To sum up, the conditions for the wormhole are $b(r) = r$ and $b' < 1$. These conditions are the same as the well-known result of Ref. [1].

5. Dynamical Ellis wormhole

The dynamical Ellis wormhole is sometimes investigated as a typical example [7, 15]. The metric is given by

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2(t)[dr^2 + (r^2 + b^2)d\Omega_2^2] \\
&= a^2(\eta)[-d\eta^2 + dr^2 + (r^2 + b^2)d\Omega_2^2] \\
&= a^2(\eta)[-dudv + (r^2 + b^2)d\Omega_2^2], \quad (49)
\end{aligned}
$$

where $a(\eta)$ is a function of the conformal time η, b is a constant and $u = \eta - r, v = \eta + r$. Note that the metric (49) is not obtained as a solution of the Einstein equation with a given matter field action as that of the Morris-Thorne wormhole.

For this spacetime, Eq. (23) becomes

$$
C_u(\dot{a}(r^2 + b^2) - r)|_S = C_v(\dot{a}(r^2 + b^2) + r)|_S, \quad (50)
$$

where $\dot{a} = da/dt$. Since $C_u, C_v > 0$, this gives us rather trivial condition

$$
r^2 < \dot{a}^2(r^2 + b^2)^2 \quad (51)
$$

at the throat candidate. The flare-out condition (24) becomes

$$
\begin{aligned}
r^a \nabla_a k|_S &= \frac{2C_u C_v \left[a\ddot{a}r^2 - \dot{a}^4(r^2 + b^2)^2 + \dot{a}^2 b^2 \right]}{a^4 \left[\dot{a}^2(r^2 + b^2)^2 - r^2 \right]}\bigg|_S \\
&> 0. \quad (52)
\end{aligned}
$$

Inequality (52) gives

$$
f(r) := -\dot{a}^4(r^2 + b^2)^2 + \dot{a}^2 b^2 + a\ddot{a}r^2 > 0. \quad (53)
$$

Using the Einstein equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = T_{\mu\nu}$ with the given metric (49), we compute the energy-momentum tensor $T_{\mu\nu}$. Then the dominant energy condition requires

$$
-T_t^t - T_r^r = \frac{2}{a^2}(a\ddot{a} + 2\dot{a}^2) \geq 0, \quad (54)
$$

$$
-T_t^t + T_\theta^\theta = \frac{2}{a^2}(-a\ddot{a} + \dot{a}^2) \geq 0, \quad (55)
$$

$$
-T_t^t + T_r^r = \frac{2}{a^2}\left(-a\ddot{a} + \dot{a}^2 - \frac{b^2}{(r^2 + b^2)^2} \right) \geq 0, (56)
$$

$$
-T_t^t - T_\theta^\theta = \frac{2}{a^2}\left(a\ddot{a} + 2\dot{a}^2 - \frac{b^2}{(r^2 + b^2)^2} \right) \geq 0, (57)
$$

where θ is the angular coordinate appeared as $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. Inequality (57) gives us stronger than that obtained from inequality (54), which gives

$$
a\ddot{a} + 2\dot{a}^2 \geq \frac{b^2}{(r^2 + b^2)^2}. \quad (58)
$$

The tightest constraint is given at $r = 0$ as

$$
a\ddot{a} + 2\dot{a}^2 \geq b^{-2}. \quad (59)
$$

In a similar way, from inequality (56), we have

$$
\dot{a}^2 - b^{-2} \geq a\ddot{a}. \quad (60)
$$

The above two inequalities imply

$$
-2\dot{a}^2 + b^{-2} \leq a\ddot{a} \leq \dot{a}^2 - b^{-2}, \quad (61)
$$

and then we see

$$
\dot{a}^2 b^2 \geq \frac{2}{3}. \quad (62)
$$

Under the energy condition (61), we can see that $f(r)$ has the maximum value at $r = 0$. Therefore, the condition for the existence of the region satisfying inequality (53) is $f(0) > 0$, which becomes

$$
\dot{a}^2 < b^{-2}. \quad (63)
$$

Using this, inequality (61) tells us

$$
-\dot{a}^2 < -2\dot{a}^2 + b^{-2} \leq a\ddot{a} \leq \dot{a}^2 - b^{-2} < 0. \quad (64)
$$

Here let us suppose $a(t)$ to be proportional to $t^{\frac{2}{3(1+w)}}$, where w is a constant. In this case, inequality (64) implies the constraint for w as

$$
-\frac{1}{3} \leq w < \frac{1}{3}. \quad (65)
$$

We can see from inequality (62) and (63) that the wormhole satisfying the dominant energy condition is realized in a certain time interval of the universe and the size is about the Hubble radius.

Note that we did not give the equation of state like $p = w\rho$ here. In the current case, w will be determined through the Einstein equation. Moreover, the energy-momentum tensor derived through the Einstein equation does not have isotropic pressure. Therefore w in inequality (65) is not directly related to the equation of state.

Setting $C_u = C_v = 1$, we see that the throat is located at $r = 0$ and the tangent of the throat orbit is obviously timelike.

6. DGP wormhole

Finally we shall consider the DGP wormhole discussed in Refs. [16, 17]. The DGP is one of the braneworld models and our four dimensional spacetime is realized as a membrane in five dimensional spacetime. In Ref. [17], Maeda et al.'s definition [7] was employed and it turned out that the spacetime on the brane has the wormhole throat. Here we reconsider the brane geometry using our current definition.

The induced metric is

$$
ds^2 = \gamma^{-2}(r)dr^2 + r^2 \left(-d\tau^2 + \cosh^2\tau d\Omega_2^2 \right), \quad (66)
$$

where

$$
\gamma^2(r) = \frac{-(r^2 - 2r_c^2) + \sqrt{r^4 - 4r_0^2 r_c^2}}{2r_c^2} \quad (67)
$$

and r_0, r_c are positive constants satisfying $r_0 > r_c$. The range of r is limited as $r \geq r_* := \sqrt{r_0^2 + r_c^2}$ so that $\gamma^2(r)$ is positive, and we see $0 \leq \gamma^2(r) < 1$.

To investigate the spacetime structure in the current scheme, it is better to introduce new coordinates $(T, \bar{R})$ defined by $T = rh(r) \sinh \tau$ and $\bar{R} = rh(r) \cosh \tau$, where

$$\ln h(r) = \int \frac{1 - \gamma}{\gamma r} dr. \tag{68}$$

Then the metric is written as

$$ds^2 = h^{-2}(r)(-dT^2 + d\bar{R}^2 + \bar{R}^2 d\Omega_2^2). \tag{69}$$

Here we choose u, v as $u = T - \bar{R}, v = T + \bar{R}$ and $a(u, v) = h^{-1}(r)$.

Now we can look at Eq. (23)

$$C_u \left[(1 - \gamma)e^\tau \cosh \tau - 1 \right]\big|_S$$
$$= -C_v \left[(1 - \gamma)e^{-\tau} \cosh \tau - 1 \right]\big|_S. \tag{70}$$

Because of $C_u, C_v > 0$, this implies

$$\gamma^2 < \tanh^2 \tau. \tag{71}$$

Note that the apparent horizon of the DGP wormhole is located at the surface satisfying $\gamma^2 = \tanh^2 \tau$.

Inequality (24) becomes

$$r^a \nabla_a k|_S$$
$$= \frac{2C_v^2 h^2 \left[\gamma^2(1 - \gamma^2) \cosh^2 \tau + r\gamma\gamma' \sinh^2 \tau \right]}{r^2 \left\{ (1 - \gamma)e^\tau \cosh \tau - 1 \right\}^2}\bigg|_S$$
$$> 0, \tag{72}$$

where $\gamma' = d\gamma(r)/dr$. Using the fact of $0 \leq \gamma^2(r) < 1$ and

$$r\gamma\gamma' = \frac{r^2}{\sqrt{r^4 - 4r_0^2 r_c^2}}(1 - \gamma^2) > 0 \tag{73}$$

derived from Eq. (67), it is easy to see that the flare-out condition (72) is always satisfied.

The traversability (25) is calculated to be

$$\alpha \Bigg[C_v^2 e^{-2\tau} \left\{ r\gamma\gamma' + \left(1 - \gamma^2 \right) \right\}$$
$$+ C_u C_v \left\{ r\gamma\gamma' - \left(1 - \gamma^2 \right) \right\} \Bigg]\Bigg|_S$$
$$= \beta \Bigg[C_u^2 e^{2\tau} \left\{ r\gamma\gamma' + \left(1 - \gamma^2 \right) \right\}$$
$$+ C_u C_v \left\{ r\gamma\gamma' - \left(1 - \gamma^2 \right) \right\} \Bigg]\Bigg|_S. \tag{74}$$

From Eq. (73), we see $r\gamma\gamma' - \left(1 - \gamma^2 \right) > 0$. This implies that α and β exist and they must be positive. Therefore, the region satisfying inequality (71) is the wormhole throat. This result is consistent with that in Ref. [17].

Setting $C_u = C_v = 1$, Eq. (70) is solved $r_S(\tau)$ as

$$r_S^2(\tau) = r_c^2(1 - \tanh^4 \tau) + r_0^2(1 - \tanh^4 \tau)^{-1}. \tag{75}$$

This is the same with result in Ref. [17].

Although one considers the vacuum brane, as stressed in Ref. [17], the energy conditions are not satisfied for the *effective* energy-momentum tensor computed from the four dimensional Einstein tensor on the brane.

IV. SUMMARY

In this paper, we proposed a new definition of the wormhole throat with the flare-out condition and the traversability for general cases in terms of the null expansion rate. This formulation is refined one of the former studies [5–7]. It can appropriately represent not only wormholes without singularities, which are mainly investigated in this field, but also the cosmological wormholes proposed in the recent work [7].

As a demonstration, we applied our formulation to several examples which include non-wormhole spacetimes too. As a result, we could confirm that our definition can work at least for the concrete examples considered here. All of our examples are spherically symmetric cases, while it is interesting to investigate whether in generic spacetimes our definition coincides with the intuitive image of wormhole. This is left for future study.

Practically interesting objects are wormholes which we can actually pass through. The dynamical Ellis wormhole is that in FLRW universe without violating any exotic matters, and thus it could exist in our Universe. However, it is too large. Because of the similar size to the Hubble radius, even if it exists, it is not observed as a compact object but rather affects to the cosmological scale physics. For actual use, small wormholes are fascinating, but it seems hard or impossible to construct such wormholes without violating the energy condition.

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