PHYSICAL REVIEW D **89,** 023008 (2014)

Relic neutrinos: Physically consistent treatment of effective number of neutrinos and neutrino mass

Jeremiah Birrell,[1] Cheng-Tao Yang,[2,3] Pisin Chen,[2,3,4] and Johann Rafelski[5]

[1]*Program in Applied Mathematics, The University of Arizona, Tucson, Arizona 85721, USA*
[2]*Department of Physics and Graduate Institute of Astrophysics, National Taiwan University, Taipei, Taiwan 10617*
[3]*Leung Center for Cosmology and Particle Astrophysics (LeCosPA), National Taiwan University, Taipei, Taiwan 10617*
[4]*Kavli Institute for Particle Astrophysics and Cosmology, SLAC National Accelerator Laboratory, Menlo Park, California 94025, USA*
[5]*Department of Physics, The University of Arizona, Tucson, Arizona 85721, USA*
(Received 10 December 2013; published 24 January 2014)

We perform a model independent study of the neutrino momentum distribution at freeze-out, treating the freeze-out temperature as a free parameter. Our results imply that measurement of neutrino reheating, as characterized by the measurement of the effective number of neutrinos N_ν, amounts to the determination of the neutrino kinetic freeze-out temperature within the context of the standard model of particle physics where the number of neutrino flavors is fixed and no other massless (fractional) particles arise. At temperatures on the order of the neutrino mass, we show how cosmic background neutrino properties, i.e., energy density, pressure, and particle density, are modified in a physically consistent way as a function of neutrino mass and N_ν.

DOI: 10.1103/PhysRevD.89.023008 PACS numbers: 51.10.+y, 95.30.Cq, 14.60.Pq, 26.35.+c

I. INTRODUCTION

The relic (i.e., background cosmic) neutrinos have not been directly measured [1,2]. Their presence and properties are inferred from reaction dynamics throughout the history of the Universe [3,4]. Important properties of the free-streaming relic neutrino background include

(1) The number of neutrino flavors, $N_\nu^f = 3$ [5].
(2) Ratio of photon to neutrino temperature T_γ/T_ν (reheating ratio).
(3) Nonthermal distortions of the neutrino distribution, which in the present work are captured by the neutrino fugacity Υ.
(4) Neutrino handedness, including effects of mass captured in terms of solution of Einstein-Vlasov equation for massive neutrinos (see Sec. IV).

For the purpose of computing the dynamics of the Universe, all of these effects (excluding mass but including neutrino handedness) can be summarized as an effective number of neutrinos, a cosmological parameter defined for relativistic neutrinos by comparing the total neutrino energy density to the energy density of a massless fermion with two degrees of freedom and standard photon reheating ratio,

$$N_\nu^c \equiv \frac{\rho_\nu}{\frac{7}{120}\pi^2((\frac{4}{11})^{1/3}T_\gamma)^4}. \tag{1}$$

We emphasize that the cosmological effective number of neutrinos is distinct from the number of neutrino flavors, $N_\nu^f = 3$, though the latter certainly would impact the former should there be any doubt about the value of N_ν^f.

The standard reheating ratio $T_\gamma/T_\nu = (11/4)^{1/3}$ implied in Eq. (1) arises from assuming that the entropy from $e^\pm$ annihilation flows solely into photons. N_ν^c is normalized such that in the simplified model where there are no nonthermal distortions and standard reheating holds, $N_\nu^c = N_\nu^f = 3$. From now on, we will refer to N_ν^c simply as N_ν while N_ν^f will be taken to be the standard model value of $N_\nu^f = 3$ [5], and from now on will be absorbed into the neutrino degeneracy factor.

As we will show, the noninteger number of neutrino degrees of freedom N_ν reported experimentally [6] can be interpreted as an effect of neutrino freeze-out and reheating. This motivates a full reexamination of the neutrino freeze-out process employing the methods developed in the context of particle freeze-out in quark-gluon plasma hadronization [7,8]. Our approach allows for us, in a model independent way, to relate the neutrino kinetic freeze-out temperature to N_ν.

Our model independent approach also allows us to derive formulas for the neutrino energy density and pressure after freeze-out as functions of both the effective number of neutrinos and the neutrino masses. These allow for a self-consistent study of the combined effects that a noninteger effective number of neutrinos and nonzero neutrino masses have on cosmological observables, a problem that, as discussed in [9], has proven difficult to approach.

The standard theory of neutrino freeze-out, based on the Einstein-Boltzmann equation with two body scattering and standard neutrino weak interactions [10–13], calculates a (small) nonthermal distortion of the neutrino distribution after freeze-out. The current state-of-the-art computation results in a slight deviation of N_ν from 3 due to the participation of the high-energy tail of the neutrino distribution in reheating, and hence a small entropy transfer from $e^\pm$ into neutrinos, together with the effect of neutrino oscillations, leading to $N_\nu^{\rm th} = 3.046$ [12].

The experimental results suggest a larger value of N_ν. The extended cosmic microwave background (CMB) data set (eCMB), baryon acoustic oscillation (BAO), and Hubble measurements (H_0), denoted by WMAP + eCMB + BAO + H_0 in Table 7 of the nine-year WMAP [14] favor an effective number of neutrinos $N_\nu = 3.55^{+0.49}_{-0.48}$. An analysis based on big bang nucleosynthesis leads to $N_\nu = 3.71^{+0.47}_{-0.45}$ [15]. Several results are presented by the Planck Collaboration [6]: Planck CMB-only data give $N_\nu = 3.36 \pm 0.34$; Planck + other CMB + BAO leads to $N_\nu = 3.30 \pm 0.27$. However, these two fits produce a 2.5 s.d. tension with direct astrophysical measurements of the Hubble constant. Including a prior on the value of H_0 (determined in supernovae and cepheids surveys), while omitting BAO information, the Planck Collaboration finds $N_\nu = 3.62 \pm 0.25$. All bounds are at the 68% confidence level.

There is currently a significant degree of interest in the precise value of N_ν, due to its impact on the spectrum of CMB fluctuations. At the current level of precision, it is certainly possible that the above measurements by these different methods agree, and agree with the theoretical result $N_\nu^{\rm th}$. Though far from definitive, these results suggest the alternate possibility that some mechanism in addition to standard two-body scattering leads to a greater entropy flow into neutrinos than predicted by standard weak interactions and hence a value of $N_\nu > 3.046$.

Several scenarios for nonstandard neutrino interactions have been investigated, including neutrino electromagnetic properties [16–24] and nonstandard neutrino electron coupling [25]. In this paper we are not proposing a new mechanism for a modified neutrino freeze-out, but rather performing a model independent analysis of the impact of a delayed freeze-out on the neutrino momentum distribution—motivated by the question of what precisely the measurement of $N_\nu = 3.62 \pm 0.25$ means for the neutrino momentum distribution.

We work under the assumption that the increase in N_ν is due to the presence of conventional but not easily identified interactions that keep neutrinos in equilibrium with the background $e^\pm$, γ plasma down to a lower temperature. In other words, we treat the kinetic freeze-out temperature, denoted by T_k, as a free parameter determined by the unknown physics and perform a parametric study of the dependence of the neutrino distribution on T_k. We show that a reduction in T_k, by whatever mechanism, leads to an increase in N_ν and is capable of achieving the values seen in the Planck data.

There are two physical effects that combine in our analysis to yield the end result:

(1) Chemical freeze-out, $T_{\rm ch}$, the temperature at which particle number changing processes such as $e^+e^- \leftrightarrow \nu\bar{\nu}$ effectively cease, and kinetic freeze-out, the temperature T_k at which all momentum exchanging processes such as $e^\pm\nu \leftrightarrow e^\pm\nu$ cease, are distinct. Once the Universe's temperature drops below the chemical freeze-out temperature $T_{\rm ch}$, there are no reactions that, in a noteworthy fashion, can change the neutrino abundance, and so particle number is conserved. However, the distribution remains in kinetic equilibrium, and hence exchanges momentum with $e^\pm$, down to $T = T_k$.

(2) Our effect requires that the temperature interval $T_k < T < T_{\rm ch}$ overlaps with $T \approx m_e$ when the electron-positron mass becomes a significant scale and reheating occurs. This allows annihilation of $e^\pm$ to feed energy and entropy into neutrinos and reduce the photon-neutrino temperature ratio T_γ/T_ν. As we will see, the freeze-out temperature for standard model neutrino scattering processes is on the border of this regime.

That "neutrino reheating" leads to an increase in N_ν is quite well known [10–12], as it is precisely this effect that leads to the standard value of $N_\nu = 3.046$. However, it is not well known that reheating of neutrinos is accompanied by an underpopulation of neutrino phase space relative to an equilibrium distribution. This underpopulation is characterized in the present context by a little-known cosmological model parameter, the neutrino fugacity Υ_ν. Its significance for neutrino cosmology has been previously recognized [26,27] but is not widely appreciated. On the other hand, in other physical processes that involve decoupling and freeze-out, such chemical parameters are in daily use as already noted [7].

Since we ask how the kinetic freeze-out T_k needs to be modified in order to explain a given value N_ν, in principle one can wonder if $T_{\rm ch}$ should also change. The general experience from other areas of physics is that it is much more difficult to find changes in $T_{\rm ch}$ beyond two-body interaction processes. The reason that T_k is more easily modified is the possible appearance of collective coherent scattering processes of the neutrino-pasmon scattering type, which add to elastic scattering and thus alter T_k but normally vanish in particle changing processes, leaving $T_{\rm ch}$ unchanged. Therefore, in our analysis, we consider the chemical freeze-out process to be fixed by standard model weak interactions. In addition to the above motivation, even if $T_{\rm ch}$ were modified, its precise value is entirely immaterial to the present study as long as $T_{\rm ch}$ occurs before $e^\pm$ annihilation begins in earnest, as demonstrated in [28].

Under this assumption, we examine the effect that a nonstandard neutrino kinetic freeze-out temperature T_k has on the form of the cosmic neutrino distribution and effective number of neutrinos after decoupling in a model independent fashion (i.e., treating T_k as a free parameter).

In Sec. II we discuss the general form of the non-equilibrium neutrino distribution, including the significance of the fugacity parameter. In II B we derive the form of the free-streaming neutrino distribution using the Einstein-Vlasov equation. In II C we compute various moments of the distribution. In Sec. III we compute the relation between neutrino fugacity, the reheating temperature ratio, and the kinetic freeze-out temperature. In Sec. III C we discuss the impact on the effective number of neutrinos. We discuss the combined impact of neutrino mass and fugacity parameters when the temperature is on the order of the neutrino mass in Sec. IV. We present our conclusions and discussion in Sec. V.

II. NONEQUILIBRIUM NEUTRINOS

A. Chemical and kinetic equilibrium

Prior to the neutrino chemical freeze-out temperature, T_{ch}, number-changing processes are significant and keep neutrinos in chemical (and thermal) equilibrium, implying that the distribution function of each neutrino flavor has the Fermi-Dirac form, obtained by maximizing entropy at fixed energy,

$$f_c(t,E) = \frac{1}{\exp(E/T)+1}, \quad \text{for } T(t) > T_{\mathrm{ch}}. \tag{2}$$

When $T_k < T(t) < T_{\mathrm{ch}}$, the number-changing process no longer occurs rapidly enough to keep the distribution in chemical equilibrium but there is still sufficient momentum exchange to keep the distribution in thermal equilibrium. The distribution function is therefore obtained by maximizing entropy, with a fixed energy, particle number, and antiparticle number separately, implying that the distribution function has the form

$$f_k(t,E) = \frac{1}{\Upsilon_\nu^{-1}\exp(E/T)+1}, \quad \text{for } T_k < T(t) < T_{\mathrm{ch}}. \tag{3}$$

The fugacity, $\Upsilon_\nu(t) \equiv e^{\sigma(t)}$, controls the occupancy of phase space and is necessary once $T(t) < T_{\mathrm{ch}}$ in order to conserve particle number. The effect of σ is similar after that of chemical potential μ, except that σ is equal for particles and antiparticles, and not opposite, as noted in [26,27]. This means $\sigma > 0$ ($\Upsilon_\nu > 1$) increases the density of both particles and antiparticles, rather than increasing one and decreasing the other as is common when the chemical potential is associated with conservation laws such as lepton number. Similarly, $\sigma < 0$ ($\Upsilon_\nu < 1$) decreases both. The fact that σ is not opposite for particles and antiparticles reflects the fact that both the number of particles and the number of antiparticles are conserved after chemical freeze-out, and not just their difference. The equality reflects the fact that any process that modifies the distribution would affect both particle and antiparticle distributions in the same fashion.

The use of $\Upsilon_\nu(t)$ to account for the processes that feed into neutrinos is nearly exact in the temperature interval after chemical and before kinetic freeze-out, since scattering processes reequilibrate the momentum distribution to this shape in order to maximize the entropy content. However, it is an approximation when the additional particle feeding occurs near kinetic freeze-out, where the energy dependence of the neutrino cross sections becomes significant, leading to an energy-dependent freeze-out and therefore additional nonthermal distortions. These could be thought of as allowing Υ_ν to be momentum dependent. In the particular case investigated in [12], these nonthermal distortions are small, below 5%. In this work, we will restrict our attention to the simplified model of a momentum-independent $\Upsilon_\nu(t)$.

B. Einstein-Vlasov equation in FRW spacetime

We begin our analysis with the simplest regime (from the neutrino perspective), $T < T_k$, when both number-changing and momentum-exchanging interactions have ceased and neutrinos begin to freely stream. The general relativistic Boltzmann equation describes the dynamics of a gas of particles that travel freely in between point interactions in an arbitrary spacetime [29–32],

$$p^\alpha \partial_{x^\alpha} f - \Gamma^j_{\mu\nu} p^\mu p^\nu \partial_{p^j} f = C[f]. \tag{4}$$

Here $\Gamma^\alpha_{\mu\nu}$ is the affine connection (Christoffel symbol), and f is a function on the mass shell,

$$g_{\alpha\beta} p^\alpha p^\beta = m^2; \tag{5}$$

hence, greek indices are summed from 0 to 3 whereas j is only summed from 1 to 3. When collisions are negligible, such as for $T < T_k$, we have $C[f] = 0$ and all particles move on geodesics, yielding the Einstein-Vlasov equation.

We now specialize to collision-free homogeneous isotropic cosmological solutions and therefore assume the flat FRW ansatz for the spacetime metric

$$g = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2). \tag{6}$$

We will make the simplifying assumption of a perfectly homogeneous universe. See [33] for a review of the results and challenges associated with the study of inhomogeneities.

Due to homogeneity and isotropy, the neutrino distribution function depends on t and $p^0 = E$ only. Therefore Eq. (4) becomes

$$E\partial_t f + (m^2 - E^2)\frac{\partial_t a}{a}\partial_E f = 0. \tag{7}$$

The general solution to Eq. (7) is known. For an example, see [31] or [33],

$$f(t,E) = K(x), \qquad x = \frac{a(t)^2}{D^2}(E^2 - m^2), \tag{8}$$

where K is an arbitrary smooth function and D is an arbitrary constant with units of mass. To continue the evolution beyond the freeze-out time, t_k, we must choose K to match at t_k the equilibrium distribution Eq. (2).

With this in mind, we let

$$K(x) = \frac{1}{\Upsilon_\nu^{-1} e^{\sqrt{x + m^2/T_k^2}} + 1} \tag{9}$$

and $D = T_k a(t_k)$ to match Eq. (3) at freeze-out. The Fermi-Dirac-Einstein-Vlasov (FDEV) distribution function for neutrinos after freeze-out is then

$$f(t,E) = \frac{1}{\Upsilon_\nu^{-1} e^{\sqrt{(E^2-m^2)/T_\nu^2 + m_\nu^2/T_k^2}} + 1}, \tag{10}$$

where

$$T_\nu(t) = \frac{T_k a(t_k)}{a(t)}. \tag{11}$$

Equation (10) provides the distribution function that describes a gas of neutrinos that have been free streaming in an expanding universe since they froze out at $T_\nu(t_k) = T_k$. We will call T_ν in Eq. (11) the neutrino background temperature, even though the distribution of free-streaming particles has a thermal shape only for $m = 0$. This language is, however, reasonable since apart from the reheating factor of photons due to e^+e^- annihilation, which we discuss in Sec. III, T_ν tracks the photon background temperature.

C. Moments of FDEV distribution

Here we compute the stress energy tensor, number current, and entropy current associated with the distribution Eq. (10),

$$\mathcal{T}^{\mu\nu} = \frac{g_\nu}{8\pi^3}\int f \frac{p^\mu p^\nu}{p_0}\sqrt{-g}d^3p, \tag{12}$$

$$n^\nu = \frac{g_\nu}{8\pi^3}\int f \frac{p^\nu}{p_0}\sqrt{-g}d^3p, \tag{13}$$

$$s^\mu = -\frac{g_\nu}{8\pi^3}\int h \frac{p^\mu}{p_0}\sqrt{-g}d^3p,$$
$$h = f\ln(f) + (1-f)\ln(1-f), \tag{14}$$

where g_ν is the neutrino degeneracy (not to be confused with the metric factor $\sqrt{-g} = a^3$). We first work with the general form of f given in Eq. (8) and later specialize to the explicit form Eq. (10).

Isotropy of the metric and of f in momentum space implies that the off-diagonal elements of the stress energy tensor and spacial components of the particle number and entropy currents vanish and that the pressure is isotropic. Hence we must compute

$$\mathcal{T}^{00} = a^3\frac{g_\nu}{8\pi^3}\int fEd^3p, \tag{15}$$

$$\mathcal{T}^{ii} = \frac{1}{3}a^3\frac{g_\nu}{8\pi^3}\int f\frac{|p|^2}{E}d^3p, \quad i = 1\ldots 3, \tag{16}$$

$$n^0 = a^3\frac{g_\nu}{8\pi^3}\int fd^3p, \tag{17}$$

$$s^0 = -a^3\frac{g_\nu}{8\pi^3}\int hd^3p, \tag{18}$$

where $|p|$ is the Euclidean norm of the spacial components of p^μ and $E = p^0$ is given by

$$m_\nu^2 = E^2 - a(t)^2|p|^2. \tag{19}$$

Computing $\mathcal{T}^{00}$ we find

$$\begin{aligned}\mathcal{T}^{00} &= \frac{g_\nu a^3}{2\pi^2}\int_0^\infty K((E^2 - m_\nu^2)/T_\nu^2)E|p|^2d|p| \\ &= \frac{g_\nu a^3}{2\pi^2}\int_0^\infty K(a^2p^2/T_\nu^2)(m_\nu^2 + a^2p^2)^{1/2}|p|^2d|p| \\ &= \frac{g_\nu}{2\pi^2}\int_0^\infty K(z^2/T_\nu^2)(m_\nu^2 + z^2)^{1/2}z^2dz,\end{aligned} \tag{20}$$

where we made a change of variables $z = a(t)|p|$. Note that z is the physically measured momentum. Similarly

$$\mathcal{T}^{ii} = \frac{g_\nu}{6\pi^2a^2}\int_0^\infty K(z^2/T_\nu^2)(m_\nu^2 + z^2)^{-1/2}z^4dz, \tag{21}$$

$$n^0 = \frac{g_\nu}{2\pi^2}\int_0^\infty K(z^2/T_\nu^2)z^2dz, \tag{22}$$

$$s^0 = -\frac{g_\nu}{2\pi^2}\int_0^\infty H(z^2/T_\nu^2)z^2dz, \tag{23}$$

$$H = K\ln K + (1-K)\ln(1-K). \tag{24}$$

We now rename z to p, so that p represents the magnitude of the physical momentum, drop the superscripts, and insert Eq. (9) for K, giving the energy density, pressure, and number density for each neutrino flavor,

$$\rho = \frac{g_\nu}{2\pi^2}\int_0^\infty \frac{(m_\nu^2+p^2)^{1/2}p^2 dp}{\Upsilon_\nu^{-1}e^{\sqrt{p^2/T_\nu^2+m_\nu^2/T_k^2}}+1}, \tag{25}$$

$$P = \frac{g_\nu}{6\pi^2}\int_0^\infty \frac{(m_\nu^2+p^2)^{-1/2}p^4 dp}{\Upsilon_\nu^{-1}e^{\sqrt{p^2/T_\nu^2+m_\nu^2/T_k^2}}+1}, \tag{26}$$

$$n = \frac{g_\nu}{2\pi^2}\int_0^\infty \frac{p^2 dp}{\Upsilon_\nu^{-1}e^{\sqrt{p^2/T_\nu^2+m_\nu^2/T_k^2}}+1}. \tag{27}$$

These differ from the corresponding expressions for an equilibrium distribution in Minkowski space by the replacement $m \to mT_\nu(t)/T_k$ *only* in the exponential.

By making a change of variables $u = p/T_\nu$, one sees that both n and s are proportional to T_ν^3. By definition, T_ν is inversely proportional to a; hence,

$$a^3 n = \text{constant} \quad \text{and} \quad a^3 s = \text{constant}. \tag{28}$$

This proves that the particle number and entropy in a comoving volume are conserved. We emphasize that this result does not depend on the particular form of K that defines the shape of the momentum distribution at freeze-out. Note further that since an eV scale or below neutrino mass is at least 6 orders of magnitude smaller than T_k, we can ignore the neutrino mass in the exponent. This remark does not extend to the energy factor multiplying the Fermi-Dirac distribution in the calculation of energy density or pressure, and therefore Eq. (25) and Eq. (26) have unusual properties, while Eq. (27) is just what one may naively expect, remembering that the mass term in this expression is completely negligible.

III. NEUTRINO FUGACITY AND PHOTON-TO-NEUTRINO TEMPERATURE RATIO

To complete our characterization of the neutrino distribution as it would have evolved from decoupling until recombination, we must understand the relation between the kinetic freeze-out temperature, photon to neutrino temperature ratio, and the fugacity. To that end, we now focus on the regime between chemical and kinetic freeze-out, $T_k < T < T_{\rm ch}$.

Before the reheating period, neutrinos, photons, electrons, and positrons have the same temperature. When the temperature approaches and drops below the electron mass, the electrons and positrons annihilate. After photon reheating, both the neutrino and photon temperatures evolve inversely proportional to a, so their ratio after reheating equals their ratio today. The resulting ratio of photon to neutrino temperatures has often been studied, but not the type of model independent study of the T_k parameter space that we present here. In the following we use subscripts 1 and 2 to denote quantities before and after reheating, respectively.

We first outline the physics of the situation qualitatively. For $T_k < T < T_{\rm ch}$, the evolution of the temperature of the common $e^\pm$, γ, ν plasma and the neutrino fugacity are determined by conservation of comoving neutrino number (since $T < T_{\rm ch}$) and conservation of entropy. The latter condition is not exactly correct once one drops the assumption of an instantaneous chemical freeze-out, but it is a very good approximation as shown in [28]. As shown in Sec. II, after thermal freeze-out the neutrinos begin to free-stream, and therefore Υ_ν is constant, the neutrino temperature evolves as $1/a$, and the comoving neutrino entropy and neutrino number are exactly conserved Eq. (28). The photon temperature then evolves to conserve the comoving entropy in photons, electrons, and positrons. As annihilation occurs, entropy from e^+e^- is fed into photons, leading to reheating. We now make this analysis quantitative in order to derive a relation between the reheating temperature ratio and neutrino fugacity.

When the (common) temperature T_1 is much larger than the electron mass and T_k, the entropy in a given comoving volume, V_1, is the sum of relativistic neutrinos (with $\Upsilon_\nu = 1$), electrons, positrons, and photons,

$$S(T_1) = \left(\frac{7}{8}g_\nu + \frac{7}{8}g_{e^\pm} + g_\gamma\right)\frac{2\pi^2}{45}T_1^3 V_1, \tag{29}$$

where T_1 is the common neutrino, e^+e^-, and γ temperature. The number of neutrinos and antineutrinos in this same volume is

$$\mathcal{N}_\nu(T_1) = \frac{3g_\nu}{4\pi^2}\zeta(3)T_1^3 V_1. \tag{30}$$

The particle-antiparticle, flavor, and spin-helicity statistical factors are $g_\nu = 6$, $g_{e^\pm} = 4$, $g_\gamma = 2$.

As discussed above, distinct chemical and thermal freeze-out temperatures lead to a nonequilibrium modification of the neutrino distribution in the form of a fugacity factor Υ_ν. This leads to the following expressions for neutrino entropy and number at $T = T_k$ in the comoving volume:

$$S(T_k) = \left(\frac{2\pi^2}{45}g_\gamma T_k^3 + S_{e^\pm}(T_k) + S_\nu(T_k)\right)V_k, \tag{31}$$

$$\mathcal{N}_\nu(T_k) = \frac{g_\nu}{2\pi^2}\int_0^\infty \frac{u^2 du}{\Upsilon_\nu^{-1}(T_k)e^u + 1}T_k^3 V_k. \tag{32}$$

After neutrino kinetic freeze-out and when $T \ll m_e$, i.e., after reheating has completed and almost all of the e^+e^- have annihilated, the entropy in neutrinos is conserved independently of the other particle species, the electron entropy is negligible, and the photon entropy is

$$S_\gamma(T_2) = \frac{2\pi^2}{45} g_\gamma T_{\gamma,2}^3 V_2. \tag{33}$$

Note that we must now distinguish between the neutrino and photon temperatures.

Conservation arguments give the following three relations:

(1) Conservation of comoving neutrino number,

$$\frac{T_1^3 V_1}{T_k^3 V_k} = \frac{2}{3\zeta(3)} \int_0^\infty \frac{u^2 du}{\Upsilon_\nu^{-1}(T_k) e^u + 1}. \tag{34}$$

(2) Conservation of $e^\pm$, γ, neutrino entropy before neutrino freeze-out,

$$\left(\frac{7}{8} g_\nu + \frac{7}{8} g_{e^\pm} + g_\gamma\right) \frac{2\pi^2}{45} T_1^3 V_1 = \left(S_\nu(T_k) + S_{e^\pm}(T_k) + \frac{2\pi^2}{45} g_\gamma T_k^3\right) V_k. \tag{35}$$

(3) Conservation of $e^\pm$, γ entropy after neutrino freeze-out (at this point, neutrino entropy is conserved independently),

$$\frac{2\pi^2}{45} g_\gamma T_{\gamma,2}^3 V_2 = \left(\frac{2\pi^2}{45} g_\gamma T_k^3 + S_{e^\pm}(T_k)\right) V_k. \tag{36}$$

These relations allow us to solve for the fugacity, reheating ratio, and effective number of neutrinos in terms of the freeze-out temperature, irrespective of the details of the dynamics that leads to a particular freeze-out temperature.

A. Neutrino fugacity

For $T \gg m_e$ the Universe is radiation dominated and $T \propto 1/a$. Therefore we can normalize the scale factor so that $T_1^3 V_1 \to 1$ as $T_1 \to \infty$. With this normalization, Eqs. (34) and (35) become two equations that can be solved numerically for $\Upsilon_\nu(T_k)$, shown in Fig. 1, and $a(T_k) = V_k^{1/3}$. We emphasize that $\Upsilon_\nu \neq 1$ is an unavoidable consequence of the freeze-out process, whenever the interval $T_k < T < T_{\mathrm{ch}}$ contains temperatures on the order of the electron mass. This latter condition is critical. If freeze-out occurs while $e^\pm$ are still effectively massless, then after setting $m/T_k = 0$ in Eq. (35), we see that $T_1^3 V_1 = T_k^3 V_k$, i.e., the temperature evolves as $T = 1/a$. Inserting this into Eq. (34) then implies that $\Upsilon_\nu = 1$. This behavior is seen in Fig. 1 when T_k/m_e is large. When $T_k/m_e = O(1)$ this argument no longer holds; there is no longer any solution with $\Upsilon_\nu = 1$. From Fig. 1 we see Υ_ν is monotonically decreasing with T_k, indicating an under-population of phase space compared to equilibrium.

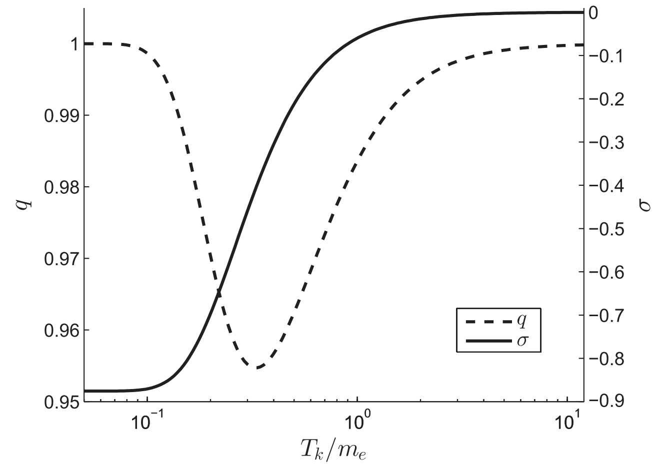

FIG. 1. Deceleration parameter (left axis) and log of neutrino fugacity (right axis) as functions of kinetic freeze-out temperature.

Figure 1 also shows the deceleration parameter

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2}\left(1 + 3\frac{P}{\rho}\right). \tag{37}$$

For a purely radiation-dominated universe, $q = 1$. A mass scale that becomes relevant at a particular temperature causes q to drop below unity, towards the matter-dominated value $q = 1/2$. $q < 1$ is therefore an indicator that conditions are right for Υ to be pushed off of its equilibrium value of unity if in addition we have $T_k < T < T_{\mathrm{ch}}$, per our discussion above. A similar plot was obtained in Ref. [28] for a more detailed model of a smooth (i.e., not instantaneous) chemical freeze-out, where we showed that the instantaneous freeze-out approximation has error of less than 1%. The result of the argument given here is equivalent to the entropy-conserving curve in that figure in Ref. [28]. Considering the current neutrino mass bounds of $O(0.1)$ eV, the values of σ achievable in a delayed freeze-out scenario are comparable to m_ν/T_r, where $T_r = 0.253$ eV is the recombination temperature. This suggests that the effects of σ may compete with the impact of neutrino mass. We will discuss this further in Sec. IV.

B. Reheating ratio

We now derive the relation between the reheating temperature ratio and neutrino fugacity. Using Eqs. (35) and (36) we can eliminate $S_{e^\pm}(T_k)$ and obtain

$$\left(\frac{7}{8} g_\nu + \frac{7}{8} g_{e^\pm} + g_\gamma\right) \frac{2\pi^2}{45} T_1^3 V_1 - S_\nu(T_k) V_k = \frac{2\pi^2}{45} g_\gamma T_{\gamma,2}^3 V_2. \tag{38}$$

Note that this by no means implies that the entropy remaining in e^+e^- at freeze-out plays no role in our discussion; it was crucial for computing $\Upsilon_\nu(T_k)$, shown in Fig. 1.

Dividing both sides of Eq. (38) by $\frac{2\pi^2}{45} g_\gamma T_k^3 V_k$ and using Eq. (34), we find

$$\frac{2}{3\zeta(3)}\left(1+\frac{7}{8}\frac{g_\nu+g_{e^\pm}}{g_\gamma}\right)\int_0^\infty \frac{u^2 du}{\Upsilon_\nu^{-1}(T_k)e^u+1} - \frac{45}{2\pi^2 g_\gamma} S_\nu(T_k)/T_k^3 = \frac{T_{\gamma,2}^3 V_2}{T_k^3 V_k}. \tag{39}$$

From Eq. (11), the neutrino temperature after kinetic freeze-out is

$$T_{\nu,2} = \frac{a(t_k)T_k}{a(t_2)} = \left(\frac{V_k T_k^3}{V_2}\right)^{1/3}. \tag{40}$$

Therefore Eq. (39) gives the photon-to-neutrino temperature ratio after freeze-out as a function of the neutrino fugacity

$$\frac{2}{3\zeta(3)}\left(1+\frac{7}{8}\frac{g_\nu+g_{e^\pm}}{g_\gamma}\right)\int_0^\infty \frac{u^2 du}{\Upsilon_\nu^{-1}e^u+1} - \frac{45}{2\pi^2 g_\gamma} S_\nu(T_k)/T_k^3 = \left(\frac{T_\gamma}{T_\nu}\right)^3. \tag{41}$$

We emphasize that $S_\nu(T_k)$ scales as T_k^3 and so $S_\nu(T_k)/T_k^3$ depends only on Υ_ν. We now write $\Upsilon_\nu = e^\sigma$ and Taylor expand $\log(T_\gamma/T_\nu)$ about $\sigma = 0$ to obtain

$$\frac{T_\gamma}{T_\nu} = a\Upsilon_\nu^b(1+c\sigma^2+O(\sigma^3)), \tag{42}$$

$$a = \left(1+\frac{7}{8}\frac{g_{e^\pm}}{g_\gamma}\right)^{1/3} = \left(\frac{11}{4}\right)^{1/3} \approx 1.4010, \tag{43}$$

$$b = \frac{\pi^2}{27\zeta(3)} \frac{1+\frac{7}{8}\frac{g_\nu+g_{e^\pm}}{g_\gamma} - \frac{3645}{8\pi^6}\zeta(3)^2\frac{g_\nu}{g_\gamma}}{1+\frac{7}{8}\frac{g_{e^\pm}}{g_\gamma}} \tag{44}$$

$$\approx 0.367, \tag{45}$$

$$c \approx -0.0209. \tag{46}$$

The second-order coefficient, c, is significantly smaller than b, making the power law approximation very accurate (to within 2% relative error) in the region of interest $.4 \le \Upsilon_\nu \le 1$. For additional precision we have included the second-order term as well, bringing the relative error down to less than 5×10^{-4} over the same range of Υ.

The above analytic discussion presents another perspective on the power law obtained in [28], using the more complex model of a smooth chemical freeze-out. There, we found the nearly identical relation

$$\Upsilon_\nu = 0.420\left(\frac{T_\gamma}{T_\nu}\right)^{2.57} \tag{47}$$

to within 1% over the region $.4 \le \Upsilon_\nu \le 1$ by a numerical fitting procedure rather than an analytic argument. For comparison with parameters a, b in Eqs. (43) and (44), this numerical approximation translates to

$$\frac{T_\gamma}{T_\nu} = \tilde{a}\Upsilon_\nu^{\tilde{b}}, \qquad \tilde{a} \approx 1.4015, \qquad \tilde{b} \approx 0.389. \tag{48}$$

C. Effective number of neutrinos

For (effectively) massless neutrinos, a deviation of the distribution function from the equilibrium form with standard reheating is summarized by the effective number of neutrinos, N_ν, defined in Eq. (1). As discussed above, the currently accepted theoretical value is $N_\nu = 3.046$ [12] after reheating, while Planck data give $N_\nu = 3.36 \pm 0.34$ (CMB only) and $N_\nu = 3.62 \pm 0.25$ (CMB + H_0) [6].

After reheating, both T_ν and T_γ evolve inversely proportional to the scale factor, and so the reheating ratio remains constant. Combining this with the fact that for massless neutrinos, ρ_ν is proportional to T_ν^4 implies $N_\nu =$ constant after reheating, at least until the temperature reaches the neutrino mass scale, at which point the definition Eq. (1) becomes inappropriate for characterizing the number of massless degrees of freedom. Even after the neutrino mass scale does become relevant, we will still use N_ν to refer to the value of the effective number of neutrinos that was established at freeze-out, even though the relation Eq. (1) will no longer hold.

Using Eq. (42) and $\Upsilon_\nu(T_k)$ from Fig. 1 we obtain T_γ/T_ν and N_ν after reheating as a function of T_k. Most importantly, note that the decrease in the reheating ratio is able to overcome the drop in phase space occupancy $\Upsilon_\nu < 1$, the combined effect being an increase in N_ν as shown in Fig. 2.

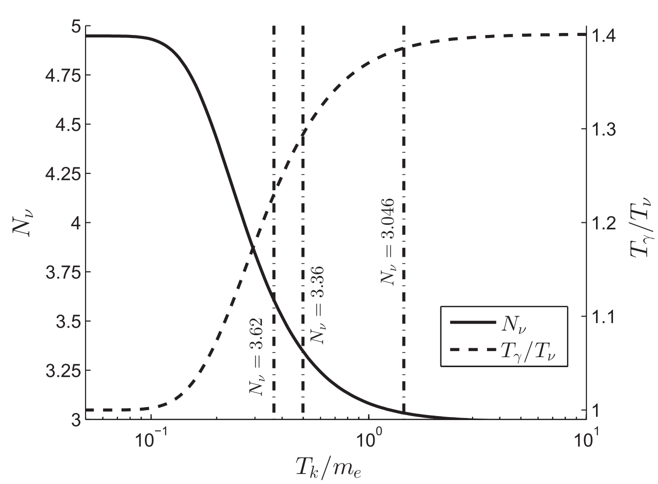

FIG. 2. Effective number of neutrinos and photon-to-neutrino temperature ratio after reheating, both as a functions of T_k.

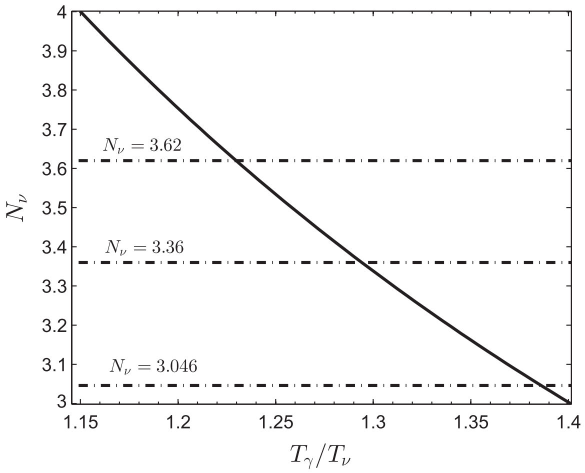

FIG. 3. N_ν after reheating, as a function of photon-to-neutrino temperature ratio.

It saturates for small T_k since both Υ_ν and the reheating ratio do.

We see that the effect of a delayed freeze-out is capable of matching a noninteger number of neutrinos in the range that is currently favored by Planck. In Fig. 2 the vertical lines indicate the value of the freeze-out temperature that corresponds to the indicated value of N_ν. A measurement of N_ν therefore is demonstrated here as being a measurement of the kinetic freeze-out temperature. Moreover, the measurement of N_ν also determines the reheating temperature ratio between photons and neutrinos, shown in the solid line in Fig. 3. Here the horizontal lines guide the eye.

Figure 4 shows the effective number of neutrinos after reheating, as a function of $\sigma = \ln \Upsilon$. Using the second-order expansion for the reheating ratio in Eq. (42), we can also present the analytic formula

$$N_\nu = \frac{360}{7\pi^4} \frac{e^{-4b\sigma}}{(1+c\sigma^2)^4} \int_0^\infty \frac{u^3}{e^{u-\sigma}+1} du (1 + O(\sigma^3)), \quad (49)$$

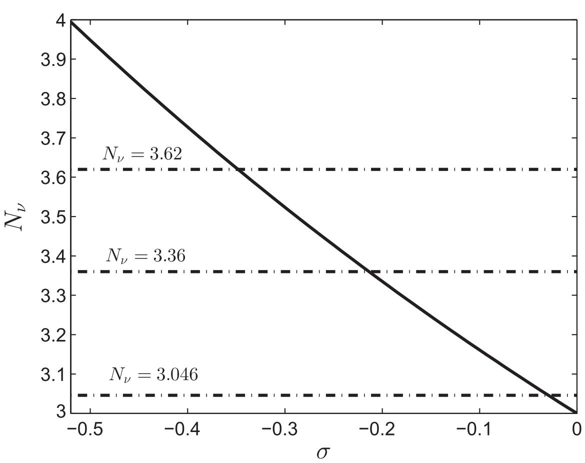

FIG. 4. Effective number of neutrinos after reheating, as a function of σ.

where a, b, c are the same as in Eq. (42). The relative error of this approximation is less than 0.002 over the range $-0.9 \le \sigma \le 0$. The second-order Taylor expansion of the integral in this expression is not sufficiently accurate over the desired range of σ, so we leave it in the presented integral-analytic form. For applications, it must be evaluated numerically.

IV. NEUTRINO MASS AND FUGACITY IN ION RECOMBINATION ERA

To this point, we have characterized the freeze-out process in terms of the kinetic freeze-out temperature T_k and obtained the form of neutrino momentum distribution that results. In Sec. II B we presented a solution of the free-streaming dynamics of neutrinos, which allows us to obtain the form of neutrino distribution at any later epoch.

The precise form of the neutrino distribution and in particular the fugacity parameter $\Upsilon_\nu = e^\sigma$ we introduced in this paper become physically relevant when the Universe's temperature approaches and drops below the neutrino mass. This is so since as long as the neutrino mass is negligible, the shape of the neutrino distribution as determined in this work has no impact on expansion dynamics of the Universe, considering that, in the absence of a scale parameter, there is no modification of the equation of state. We now characterize the regime when the mass of neutrinos becomes a relevant scale including the effects of fugacity Υ_ν.

To compare the energy density Eq. (25) and pressure Eq. (26) to that of a massless particle distribution with $\Upsilon = 1$, we make a change of variables $u = p/T_\nu$ and neglect terms involving $m/T_k \ll 1$,

$$\rho^{\mathrm{EV}} \simeq \frac{g_\nu T_\nu^4}{2\pi^2} \int_0^\infty \frac{(m_\nu^2/T_\nu^2 + u^2)^{1/2} u^2}{\Upsilon^{-1} \exp(u) + 1} du, \quad (50)$$

$$P^{\mathrm{EV}} \simeq \frac{g_\nu T_\nu^4}{6\pi^2} \int_0^\infty \frac{(m_\nu^2/T_\nu^2 + u^2)^{-1/2} u^4}{\Upsilon^{-1} \exp(u) + 1} du, \quad (51)$$

where the upper index "EV" reminds us that we have used the Einstein-Vlasov free-streaming solution for the neutrino distribution. For $m \ll T_\nu$ the massless equation of state $\rho^{\mathrm{EV}} = 3P^{\mathrm{EV}}$ holds, but when T_ν is on the order of the mass, the mass term becomes important and modifies the equation of state. The lack of a mass term in the exponential gives this a distinctly different behavior from the equilibrium Fermi-Dirac distribution.

To illustrate the effect of fugacity and neutrino mass on the equation of state, we examine the energy density and pressure of the neutrino distribution. We separate off the zero mass, $\Upsilon_\nu = 1$ contributions from a single neutrino flavor with standard reheating by defining

$$\rho_0 = \frac{7\pi^2}{120}\left[\left(\frac{4}{11}\right)^{1/3} T_\gamma\right]^4, \qquad P_0 = \rho_0/3. \tag{52}$$

We note that $\rho^{\rm EV}/\rho_0$ and $P^{\rm EV}/P_0$ are functions of $(m/T_\gamma)^2$, and N_ν, where the latter dependence is obtained by inverting both $\Upsilon(T_\gamma/T_\nu)$ from Eq. (42) and $N_\nu(\Upsilon)$ from Eq. (49) in order to obtain $T_\gamma/T_\nu(\Upsilon)$ and $\Upsilon(N_\nu)$. In practice, these inversions are best done numerically. Thus the quantities of interest are $\rho^{\rm EV}/\rho_0$ from Eq. (50) and Eq. (52) and the corresponding expressions for the pressure, both as functions of $\delta N_\nu = N_\nu - 3$ and $\beta = m_\nu/T_\gamma$. Again, we emphasize that here N_ν refers to the value of the effective number of neutrinos that was established at neutrino freeze-out when neutrinos were still effectively massless. At the temperatures we are now considering, Eq. (1) no longer applies.

The functional dependence of the energy density and pressure that we find is best characterized by a simple polynomial representation that arises from a least squares fit,

$$\rho^{\rm EV}/\rho_0 = N_\nu + 0.1016\sum_i \beta_i^2 + 0.0015\delta N_\nu \sum_i \beta_i^2 - 0.0001\delta N_\nu^2 \sum_i \beta_i^2 - 0.0022\sum_i \beta_i^4, \tag{53}$$

$$P^{\rm EV}/P_0 = N_\nu - 0.0616\sum_i \beta_i^2 - 0.0049\delta N_\nu \sum_i \beta_i^2 + 0.0005\delta N_\nu^2 \sum_i \beta_i^2 + 0.0022\sum_i \beta_i^4. \tag{54}$$

For a single massive neutrino, these fits are valid to within 2% and 5% relative error, respectively, in the region $3 \le N_\nu \le 5$, $0 \le m_\nu/T_\gamma \le 4$. Note that the upper limit is sufficient to cover the era of electron-ion recombination at $T_\gamma = O(0.3)$ eV and neutrino masses less than the upper bound $\sum m_i < 0.23$ eV reported in [6].

For small mass-to-temperature ratios, the fit depends only on the sum of neutrino masses squared, $\sum_i \beta_i^2 \equiv (\sum_i m_i^2)/T_\gamma^2$. However, the fourth-order term is not negligible over the chosen fitting region. Removing that term results in a maximum relative error between the fit and the exact result of greater than 25%. This indicates that one may be able to use a fit to cosmic data to constrain the hierarchy structure of the neutrino mass spectrum using the fitted values of $\sum_i m_i^2$ and $\sum_i m_i^4$.

The formulas in Eq. (53), or the more precise quantities Eqs. (50) and (42) that we fit in order to obtain them, should be used when exploring the combined effects of $N_\nu \neq 3$ and neutrino mass on cosmological observables as they properly capture the interplay between neutrino mass and the shape of the neutrino distribution in terms of physical observables. In particular, they can be used to extract fits to $\sum_i m_i^2$ and $\sum_i m_i^4$ while separating off the effects of the confounding variable N_ν.

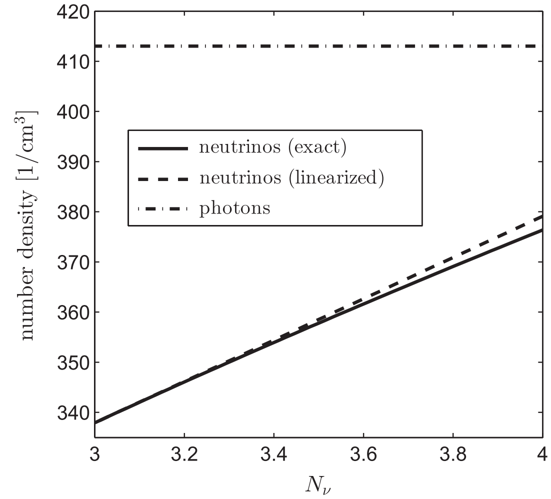

FIG. 5. Total neutrino and antineutrino number density in the Universe "today" as a function of N_ν, both exact (solid) and linearized (dashed). The photon density (dot-dashed) is also shown as a reference.

Similarly we find the net number of neutrinos after freeze-out as a function of the effective number of neutrinos N_ν. Using Eqs. (27), (42), and (49) to Taylor expand the neutrino number density as a function of δN_ν, we find

$$n_{\rm today} = (0.1993 + 0.02429\delta N_\nu) T_\gamma^3. \tag{55}$$

In the present-day Universe, at $T_\gamma = T_{\rm CMB} = 0.2349$ meV, we show both the exact and linearized results in Fig. 5. We note that at $\delta N_\nu = 0$ we find 112.6 cm^{-3} per flavor, in close agreement with [33].

V. DISCUSSION

The cosmic neutrino momentum distribution is a critical input into our understanding of the spectrum of CMB fluctuations and arguably its understanding is a prerequisite for the consideration of cosmic neutrino detection opportunities. Motivated by hints of a tension between the Planck results and the standard theory of neutrino freeze-out expressed by a noticeably noninteger value of the effective number of neutrinos N_ν, we have undertaken a model independent study of the effect of a delayed kinetic freeze-out on N_ν, and more generally on the form of the neutrino momentum distribution. The search for reaction mechanisms that can produce a reduction in the neutrino freeze-out temperature T_k from a value near $T_k/m_e \simeq 1.1$ to a value perhaps as small as $T_k/m_e \simeq 0.35$ (see Fig. 2) is a topic for future investigation.

Possible participation of neutrinos in $e^\pm$ annihilation reheating and hence a ratio of photon-to-neutrino temperature that is closer to one, and thus $N_\nu > 3$, is a well-known fact [12]. However, less appreciated is the impact of neutrino reheating on neutrino fugacity Υ_ν, the factor describing the neutrino distribution compared to chemical equilibrium $\Upsilon_\nu = 1$.

We have shown how an increase in N_ν is naturally interpreted to be due to a delayed neutrino kinetic freeze-out temperature T_k. We derived an approximate power law relation, Eq. (42), between the fugacity factor and the photon to neutrino temperature ratio that arises from a delayed freeze-out. We found a fugacity Υ_ν less than unity, and thus an underpopulation of phase space compared to chemical equilibrium.

After freeze-out, neutrinos freely stream through the expanding universe. The nonthermal modification from the fugacity factor is frozen into the shape of the distribution. We derived how this modified neutrino distribution evolves as the Universe expands and explored how the energy density and pressure are modified, including in this study the interplay of fugacity and neutrino mass. We note that these effects impact the cosmological study of the question of neutrino mass hierarchy: the effects we present in Eqs. (53) and (54) produce a functional dependence on both $\sum_i m_i^2$ and $\sum_i m_i^4$.

The fits Eqs. (53) and (54) of the exact but intractable Eqs. (50) and (51) show how the energy density and pressure are self-consistently modified when both neutrino mass and a noninteger N_ν are present. The latter dependence is expressed by using $T_\gamma/T_\nu(N_\nu)$ and $\Upsilon_\nu(N_\nu)$ in Eqs. (50) and (51) as discussed in Sec. IV. This polynomial presentation of the cosmic neutrino energy density and pressure resolves an old problem of cosmic neutrino physics by allowing a physically consistent treatment of the combined effects of neutrino mass and δN_ν as long as the magnitude of δN_ν follows from neutrino freeze-out dynamics.

ACKNOWLEDGMENTS

This work has been supported by a grant from the U.S. Department of Energy, No. DE-FG02-04ER41318 and by the Department of Defense (DoD) through the National Defense Science & Engineering Graduate Fellowship (NDSEG) Program.

[1] A. Ringwald, Nucl. Phys. **A827**, 501c (2009); arXiv:hep-ph/0301157.
[2] G. Tupper, M. Danos, B. Muller, and J. Rafelski, Phys. Rev. D **35**, 394 (1987).
[3] A. D. Dolgov, Phys. Rep. **370**, 333 (2002).
[4] J. Lesgourgues and S. Pastor, Phys. Rep. **429**, 307 (2006).
[5] ALEPH, DELPHI, L3, OPAL, and SLD Collaborations, LEP Electroweak Working Group, SLD Electroweak Group, and SLD Heavy Flavour Group, Phys. Rep. **427**, 257 (2006).
[6] Planck Collaboration, arXiv:1303.5076.
[7] J. Letessier and J. Rafelski, Eur. Phys. J. A **35**, 221 (2008).
[8] I. Kuznetsova and J. Rafelski, Eur. Phys. J. C **51**, 113 (2007).
[9] J. Lesgourgues, G. Mangano, G. Miele, and S. Pastor, *Neutrino Cosmology* (Cambridge University Press, Cambridge, England, 2013).
[10] R. E. Lopez, S. Dodelson, A. Heckler, and M. S. Turner, Phys. Rev. Lett. **82**, 3952 (1999).
[11] N. Y. Gnedin and O. Y. Gnedin, Astrophys. J. **509**, 11 (1998).
[12] G. Mangano, G. Miele, S. Pastor, T. Pinto, O. Pisanti, and P. D. Serpico, Nucl. Phys. **B729**, 221 (2005).
[13] S. Hannestad and J. Madsen, Phys. Rev. D **52**, 1764 (1995).
[14] G. Hinshaw, D. Larson, E. Komatsu, D. N. Spergel, C. L. Bennett *et al.*, Astrophys. J. Suppl. Ser. **208**, 19 (2013).
[15] G. Steigman, Adv. High Energy Phys. **2012**, 268321 (2012).
[16] J. Morgan, Phys. Lett. **102B**, 247 (1981).
[17] P. Elmfors, K. Enqvist, G. Raffelt, and G. Sigl, Nucl. Phys. **B503**, 3 (1997).
[18] P. Vogel and J. Engel, Phys. Rev. D **39**, 3378 (1989).
[19] M. Fukugita and S. Yazaki, Phys. Rev. D **36**, 3817 (1987).
[20] M. Deniz *et al.* (TEXONO Collaboration), Phys. Rev. D **81**, 072001 (2010).
[21] C. Broggini, C. Giunti, and A. Studenikin, Adv. High Energy Phys. **2012**, 459526 (2012).
[22] J. Barranco, O. Miranda, and T. Rashba, Phys. Lett. B **662**, 431 (2008).
[23] M. Hirsch, E. Nardi, and D. Restrepo, Phys. Rev. D **67**, 033005 (2003).
[24] N. Tanimoto, I. Nakano, and M. Sakuda, Phys. Lett. B **478**, 1 (2000).
[25] G. Mangano, G. Miele, S. Pastor, T. Pinto, O. Pisanti, and P. D. Serpico, Nucl. Phys. **B756**, 100 (2006).
[26] J. Bernstein, L. Brown, and G. Feinberg, Phys. Rev. D **32**, 3261 (1985).
[27] A. Dolgov and K. Kainulainen, Nucl. Phys. **B402**, 349 (1993).
[28] J. Birrell, C.-T. Yang, P. Chen, and J. Rafelski, Mod. Phys. Lett. A **28**, 1350188 (2013).
[29] H. Andreasson, Living Rev. Relativity **14**, 4 (2011).
[30] C. Cercignani and G. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhuser Verlag, Basel, Switzerland, 2000).
[31] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Oxford University Press, Oxford, England, 2009).
[32] J. Ehlers, in *Relativity, Astrophysics and Cosmology*, (D. Reidel Publishing Company, Dordrecht, Netherlands, 1973), p. 1.
[33] Y. Wong, Annu. Rev. Nucl. Part. Sci. **61**, 69 (2011).