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Parity-violating quantum kinetic theory in (2 + 1) dimensions

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We study the kinetic theory for a (2 + 1)-dimensional fermionic system with special emphasis on the parity violating properties associated with the fermion mass. The Wigner function approach is used to derive hydrodynamical transport coefficients to the first spatial derivative order. As a first attempt, the collisions between fermions are neglected. The resulting system is dissipationless. The parity violating Hall electric conductivity has the same temperature and chemical potential dependence as the quantum field theory result at one loop. Vorticity dependent transport properties, which were not considered before, also emerge naturally in this approach.

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I. INTRODUCTION

Recently an asymmetry of certain charge dependent azimuthal correlations have been observed by the STAR [1,2] and PHENIX experiments [3] at the Relativistic Heavy-Ion Collider (RHIC) and by the ALICE experiment [4] at the Large Hadron Collider (LHC). Such an asymmetry disappears at low energies where the chiral symmetry is broken [5]. One possible explanation of this phenomenon is the Chiral Magnetic Effect (CME) [6–8]. The CME and related topics have been studied in several approaches, including AdS/CFT correspondence [9–13], relativistic hydrodynamics [14–17], kinetic theory [18–23], lattice calculations [24–28] and approaches from field theory [7,29–40]; for a recent review, see, e.g., [41].

Different from the (3 + 1) dimensions [(3 + 1)D], the mass term in the Dirac equation in (2 + 1) dimensions [(2 + 1)D] explicitly breaks the parity [42]. After integration over the fermionic degrees of freedom, a Chern-Simons (CS) term $\propto \epsilon^{\sigma\rho\alpha}A_\sigma F_{\rho\alpha}$, with A_σ an Abelian gauge field, is induced in the effective action S_{eff} from the one loop correction. The fermion number current j_v^σ can then be obtained by taking the functional derivative of S_{eff} with respect to A_σ [43–47],

$$j_v^\sigma = \frac{i}{Q}\frac{\delta S_{\text{eff}}}{\delta A^\sigma} = -\text{sign}(m)\frac{Q}{8\pi}\epsilon^{\sigma\rho\alpha}F_{\rho\alpha}, \tag{1}$$

where m and Q are the mass and charge of the fermion, respectively, and sign(x) is a sign function denoting the sign of x. Therefore, the fermion number density j_v^0 is modified by the magnetic field, and the spatial components show a behavior of Hall conductivity $j^1 \propto E^2$ and $j^2 \propto E^1$. In the massless case, parity is conserved classically but broken quantum mechanically (parity anomaly). The one loop effective action has an ultraviolet divergence. To regularize this divergence, one can use the standard Pauli-Villars regularization method which preserves gauge symmetry,

$$S_{\text{eff}}^{\text{reg}}[m=0] = S_{\text{eff}}[m=0] - \lim_{M\to\infty} S_{\text{eff}}[M], \tag{2}$$

where M is a hypothetical mass and plays the role of a cutoff. However, this cutoff term will induce a CS term just like in the massive case which breaks parity [44,45]. In a non-Abelian gauge field theory, the CS term proportional to $\epsilon^{\sigma\rho\alpha}\text{Tr}[A_\sigma\partial_\rho A_\alpha + \frac{2}{3}A_\sigma A_\rho A_\alpha]$ breaks parity, the prefactor of this non-Abelian CS term needs to be quantized to preserve gauge symmetry under a large gauge transformation [48]. There are subtle issues about how to preserve this gauge invariance at non-Abelian theories at finite temperatures. Here we just work in the Abelian case and compare our results to quantum field theory calculations at one loop. At finite chemical potential μ and zero temperature, if $\mu^2 < m^2$, there is no dependence on μ [49,50] and the current returns to Eq. (1). If $\mu^2 > m^2$, the Chern-Simons term vanishes [51]. For recent reviews about the Chern-Simons theory, see, e.g., Refs. [48,50].

Parity violating hydrodynamics in (2 + 1)D with Hall viscosity has recently drawn a lot of attention in effective and holographic theories [52–58], as well as in condensed matter physics [59–63]. A systematic discussion about the constraints from the second law of thermodynamics in (2 + 1)D relativistic hydrodynamics is given by Ref. [64]. Similar constraints can be derived in a curved space [65,66]. The equality constraints on (2 + 1)D parity violating hydrodynamics are derived in Ref. [67] from general properties of Euclidean field theory in equilibria with slowly varying background metrics and gauge fields.

In previous works by some of us [18,22], a quantum kinetic theory was proposed to describe the CME and the Chiral Vortical Effect (CVE) through the gauge invariant

Wigner function [18]. The $U(1)$ and $U(1)_A$ currents induced by magnetic field and vorticity were obtained from the vector and axial-vector components of the Wigner function. The axial $U(1)_A$ current led to a local polarization effect along the vorticity direction in peripheral heavy ion collisions. A chiral kinetic equation was also derived from the Wigner function with features of the Berry phase and monopole [22].

In this paper, we try to extend our previous works of quantum kinetic theory in (3 + 1)D to (2 + 1)D. As a first attempt, we turn off interactions among fermions and assume a constant electromagnetic background field $F^{\mu\nu}$ counted in the same order as the hydrodynamic scale. Then we expand the equations of Wigner functions in powers of the Knudsen number K, defined as the ratio of the mean free path to the macroscopic and hydrodynamic scale, which is equivalent to gradient expansion. In the zeroth order, we reproduce all macroscopic quantities as well as equations of motion in an ideal fluid. In the first order, the electromagnetic field and vorticity appear in the current and energy-momentum tensor. To test the self-consistency of our approach, we compute constraints up to the second order. The energy momentum tensor, the fermion number current, and the entropy current with parity violating terms can be obtained from the Wigner function after integrating over the 3-momenta. At zero temperature and zero chemical potential, the current induced by a magnetic field is consistent with Eq. (1) in Chern-Simons theory. The current induced by vorticity is also obtained. We finally prove that the entropy is conserved.

The paper is organized as follows. In Sec. II, we give basic properties of Dirac matrices and parity for fermionic fields in (2 + 1)D. Section III is devoted to quantum kinetic theory in (2 + 1)D. In Sec. IV we solve the Winger function order by order in the space-time derivative expansion. In Sec. V, the energy-momentum tensor, the fermion number, and entropy current are obtained by momentum integration. The Hall and vorticity terms are reproduced. The Landau frame is discussed in Sec. VI. Finally we present a summary and conclusion in Sec. VII.

Our conventions and notations are $g^{\mu\nu}=\mathrm{diag}\{+,-,-\}$, $u^\mu u_\mu = 1$, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, with $u^\mu(x)$ the fluid velocity. For a vector h^μ, it can be decomposed as $h^\mu = (u\cdot h)u^\mu + \bar{h}^\mu$, with $\bar{h}^\mu = \Delta^{\mu\lambda}h_\lambda$. We also define the comoving derivative of a space-time quantity a as $\dot{a} = da/dt = u^\mu \partial_\mu a$.

II. DIRAC γ MATRICES AND PARITY IN (2 + 1)D

We choose the representation of the γ matrices as follows:

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\gamma^2 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3}$$

where σ_i are Pauli matrices. The γ matrices satisfy

$$\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}, \qquad [\gamma_\alpha, \gamma_\beta] = 2i\epsilon_{\alpha\beta\rho}\gamma^\rho$$
$$\gamma_\alpha\gamma_\beta = g_{\alpha\beta} + i\epsilon_{\alpha\beta\rho}\gamma^\rho, \tag{4}$$

where $\epsilon^{\alpha\beta\rho}$ is the Levi-Civita (antisymmetric) tensor with $\epsilon^{012} = \epsilon_{012} = 1$. The Hermitian conjugation is $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Note that there is no chirality in (2 + 1)D due to the absence of γ^5 since $i\gamma^0\gamma^1\gamma^2 = -1$. Note that the Dirac gamma matrices in Eq. (3) provide one of the irreducible representations. One can immediately write down another irreducible representation by flipping the sign of γ^σ. The nature of the lowest Landau level is different for two representations; e.g., if γ^σ corresponds to the positive energy state $E_0 = m$, then $-\gamma^\sigma$ corresponds to the negative energy state $E_0 = -m$. In real (2 + 1)D relativistic systems, such as graphene, one uses a reducible representation which combines these two irreducible representations (see, e.g., Ref. [68]).

The parity transformation in (2 + 1)D is defined by flipping the sign of one spatial component of a vector, for instance, $\mathbf{x} \to \tilde{\mathbf{x}}$ where $\tilde{\mathbf{x}} = (-x^1, x^2)$, and then a spinor transform as $\psi(t, \mathbf{x}) \to \gamma^1\psi(t, \tilde{\mathbf{x}})$. So we see that the mass term transforms as $\bar{\psi}(t, \mathbf{x})\psi(t, \mathbf{x}) \to -\bar{\psi}(t, \tilde{\mathbf{x}})\psi(t, \tilde{\mathbf{x}})$, which is a pseudoscalar.

III. QUANTUM KINETIC EQUATION FOR WIGNER FUNCTION

In a quantum kinetic approach, we replace the phase-space distribution $f(x, p)$ by the Wigner function $W(x, p)$ with the space-time position x and the (2 + 1) momentum p. It is defined as the ensemble average of the gauge invariant Wigner operator [69–71]. For fermions with mass m and charge Q, the (α, β) component of the Wigner function operator can be written as

$$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^3y}{(2\pi)^3} e^{-ip\cdot y}\bar{\psi}_\beta(x+y/2)U(x,y)\psi_\alpha(x-y/2), \tag{5}$$

where the gauge link $U(x, y)$ ensures the gauge invariance of $\hat{W}_{\alpha\beta}$ and is defined by

$$U(x, y) \equiv \exp\left[-iQ\int_{x-y/2}^{x+y/2} dz^\mu A_\mu(z)\right]. \tag{6}$$

As a first attempt, we consider a system of collisionless fermions in a constant external electromagnetic field $F_{\mu\nu}$. In this case, we can drop the path ordering in the gauge link in Eq. (6).

In quantum field theory, physical quantities correspond to matrix elements of operators under the time ordering ($\mathcal{T}$). This is also true for the Wigner function. Using

$$\mathcal{T}\hat{W} = :\hat{W}: + \langle 0|\mathcal{T}\hat{W}|0\rangle, \tag{7}$$

where the colons : : indicate a normal ordering, we can isolate the medium effect which comes from the ensemble average of $:\hat{W}:$ with a constant electromagnetic background, and the vacuum (meaning the zero chemical potential and zero temperature ground state) contribution $\langle 0|\mathcal{T}\hat{W}|0\rangle$.

When the vacuum contribution vanishes, one can simply use the normal ordering part as was done in Refs. [18,22,69–71]. For example, in (3 + 1)D, one cannot write down a nonvanishing and gauge invariant expression for the matrix element $\langle 0|\mathcal{T}\{\mathrm{Tr}[\hat{W}\gamma^\mu]\}|0\rangle$ using combinations of $F^{\mu\nu}$, D_μ, $g^{\mu\nu}$, and $\epsilon^{\mu\nu\alpha\beta}$. The simplest combination $\partial_\mu F^{\mu\nu}$ vanishes in vacuum. Analogously, one cannot write down a nonvanishing contribution for the axial current $\langle 0|\mathcal{T}\{\mathrm{Tr}[\hat{W}\gamma^\mu\gamma^5]\}|0\rangle$ either. The simplest choice $\epsilon^{\mu\nu\alpha\beta}(\partial_\nu + iQA_\nu)F_{\alpha\beta} = iQ\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}$ is gauge dependent. Therefore, there will be no vacuum contribution to these currents. This argument is consistent with the fact that all CME and CVE coefficients vanish at zero temperature, charge, and chiral chemical potential limit. In (2 + 1)D, however, we have a vacuum contribution to the vector current as shown in Eq. (1). Thus, we need to associate the Wigner distribution function to the ensemble average of $\mathcal{T}\hat{W}$.

From the equation of motion of a Dirac field, the master equation for the Winger operator is obtained [69–71]:

$$\gamma^\mu\left(p^\mu + \frac{1}{2}i\nabla^\mu - m\right)\mathcal{T}\hat{W}(x,p) = 0, \tag{8}$$

where $\nabla^\mu \equiv \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$. The vacuum matrix element of this operator equation implies $\langle 0|\mathcal{T}\hat{W}|0\rangle$ satisfies the same equation as well. Then by Eq. (7), $:\hat{W}:$ and its ensemble average, $W \equiv \langle :\hat{W}:\rangle$, also satisfy the same equation,

$$\gamma^\mu\left(p^\mu + \frac{1}{2}i\nabla^\mu - m\right)W(x,p) = 0. \tag{9}$$

Although the medium effect can be derived from the kinetic theory based on Eq. (9), this approach is not restrictive enough to fully determine the vacuum contributions. Thus, we will just match them to the quantum field theory results and combine them with the medium contributions later.

The Winger function W can be expanded in terms of four independent generators $\{1, \gamma^\sigma\}$ of the Clifford algebra,

$$W = \frac{1}{2}(\mathcal{F} + \gamma^\sigma \mathcal{V}_\sigma), \tag{10}$$

where

$$\mathcal{F} \equiv \mathrm{Tr}[W], \qquad \mathcal{V}^\sigma \equiv \mathrm{Tr}[\gamma^\sigma W]. \tag{11}$$

We can obtain the medium part of the fermion number current j^σ from $\mathcal{V}^\sigma$ by integration over p,

$$j_m^\sigma = \int d^3p\,\mathcal{V}^\sigma = \int d^3p\,\mathrm{Tr}[\gamma^\sigma W]. \tag{12}$$

The total fermion number current j^σ is the sum of the medium and vacuum part,

$$j^\sigma = j_m^\sigma + j_v^\sigma, \tag{13}$$

where j_v^σ is given by Eq. (1). The energy-momentum tensor can also be obtained from $\mathcal{V}^\sigma$,

$$T^{\sigma\rho} = \frac{1}{2}\int d^3p\,p^{(\sigma}\mathcal{V}^{\rho)} = \frac{1}{2}\int d^3p\,p^{(\sigma}\mathrm{Tr}[\gamma^{\rho)}W], \tag{14}$$

where the parentheses denote index symmetrization.

Substituting Eq. (10) into Eq. (9) yields

$$p\cdot\mathcal{V} - m\mathcal{F} = 0, \tag{15}$$

$$\nabla\cdot\mathcal{V} = 0, \tag{16}$$

$$p^\sigma\mathcal{F} - \frac{1}{2}\epsilon^{\sigma\lambda\rho}\nabla_\lambda\mathcal{V}_\rho - m\mathcal{V}^\sigma = 0, \tag{17}$$

$$\frac{1}{2}\nabla^\sigma\mathcal{F} + \epsilon^{\sigma\lambda\rho}p_\lambda\mathcal{V}_\rho = 0. \tag{18}$$

In Eqs. (15)–(18), there are eight highly consistent equations for four components of $\mathcal{F}$ and $\mathcal{V}^\sigma$.

IV. SOLVING WINGER FUNCTION IN EXPANSION OF SPACE-TIME DERIVATIVES

Since we are interested in long wavelength physics, we set up the system near equilibrium so that we can expand $\mathcal{F}$ and $\mathcal{V}^\sigma$ in powers of the space-time derivative ∂_x. We also assume that the background field $F^{\rho\sigma} = \partial^\rho A^\sigma - \partial^\sigma A^\rho$ is of the same order as ∂_x. In this case, $\mathcal{V}^\sigma$ and $\mathcal{F}$ can be written as

$$\mathcal{V}^\sigma = \mathcal{V}_{(0)}^\sigma + \mathcal{V}_{(1)}^\sigma + O(\partial_x^2), \tag{19}$$

$$\mathcal{F} = \mathcal{F}_{(0)} + \mathcal{F}_{(1)} + O(\partial_x^2), \tag{20}$$

where the indices (0), (1) denote the zeroth and the first order respectively. From Eqs. (17) and (18), we see that $\mathcal{V}_{(n)}^\sigma$ is related to $\mathcal{F}_{(n-1)}$ and that $\mathcal{F}_{(n)}$ is related to $\mathcal{V}_{(n-1)}^\sigma$. So we can solve $\mathcal{V}^\sigma$ and $\mathcal{F}$ order by order.

A. Zeroth order

At the zeroth order, Eqs. (15)–(18) become

$$p\cdot\mathcal{V}_{(0)} - m\mathcal{F}_{(0)} = 0, \tag{21}$$

$$p^\sigma\mathcal{F}_{(0)} - m\mathcal{V}_{(0)}^\sigma = 0, \tag{22}$$

$$\epsilon^{\sigma\lambda\rho}p_\lambda\mathcal{V}_\rho^{(0)} = 0. \tag{23}$$

The general forms for $\mathcal{V}^{\mu}_{(0)}$ and $\mathcal{F}$ satisfying the above equations are

$$\mathcal{V}^{\sigma}_{(0)} = p^{\sigma} V \delta(p^2 - m^2), \tag{24}$$

$$\mathcal{F}_{(0)} = mV\delta(p^2 - m^2), \tag{25}$$

where V is a function of x and p.

As we mentioned before, we set up a perturbative scheme around the equilibrium state. Therefore, the zeroth order should correspond to an equilibrium noninteracting ideal gas, where the macroscopic quantities can be obtained by the Fermi-Dirac distribution. On the other hand, these quantities can also be obtained from $\mathcal{V}^{\sigma}_{(0)}$ or $\mathcal{F}_{(0)}$. We set V to be

$$V = \frac{1}{2\pi^2} \sum_{e=\pm} \frac{1}{e^{e(u\cdot p-\mu)/T}+1} \theta(eu\cdot p - |m|), \tag{26}$$

where $e = \pm$ denote fermions/antifermions, u^{σ} denotes the fluid velocity, T is the temperature, μ is the chemical potential, and $\theta(x) = 1,\ 0$ for positive/negative x with $\theta(0) = 1/2$. Note that V becomes a space-time dependent function via the dependence on $\mu(x)$, $T(x)$, and $u^{\sigma}(x)$. Integrating $\mathcal{V}^{\sigma}_{(0)}$ over the (2 + 1) momentum, we obtain the current at the leading order,

$$\begin{aligned} j^{\sigma}_{m(0)} &= nu^{\sigma} \\ n &= \frac{1}{2\pi}\int_{|m|}^{\infty} dE_p E_p \left[\frac{1}{e^{\beta(E_p-\mu)}+1} - \frac{1}{e^{\beta(E_p+\mu)}+1}\right], \end{aligned} \tag{27}$$

where n is the fermion number density at the zeroth order. Note that the contribution from antifermions is negative. From Eq. (14), we also obtain $T^{\sigma\rho}$ at the zeroth order,

$$\begin{aligned} T^{\sigma\rho}_{m(0)} &= (\varepsilon + P)u^{\sigma}u^{\rho} - Pg^{\sigma\rho} \\ \varepsilon &= \frac{1}{2\pi}\int_{|m|}^{\infty} dE_p E_p^2 \left[\frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1}\right], \\ P &= \frac{1}{4\pi}\int_{|m|}^{\infty} dE_p (E_p^2 - m^2) \\ &\quad \times \left[\frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1}\right], \end{aligned} \tag{28}$$

where ε and P are energy density and pressure, respectively. Note that the contribution from antifermions is positive.

The θ function in Eq. (26) is added such that the kinetic theory gives the correct free fermion contribution to j^{σ} and $T^{\sigma\rho}$ at the leading order, with fermions and antifermions contributing by a different sign to the fermion number density n in Eq. (27) but the same sign to the energy density ε and the pressure P in Eq. (28) (see Appendix C for detailed derivations). Without the θ function, one could only get one but not both of the signs right. The form of the θ function, however, is not unique. For example, one can use $\theta(eu \cdot p)$ instead of $\theta(eu\cdot p - |m|)$ without changing the physics. We also demonstrate this in Appendix C.

B. First order

At the first order, Eqs. (15)–(18) become

$$p \cdot \mathcal{V}_{(1)} - m\mathcal{F}_{(1)} = 0, \tag{29}$$

$$\nabla \cdot \mathcal{V}_{(0)} = 0, \tag{30}$$

$$p^{\sigma}\mathcal{F}_{(1)} - \frac{1}{2}\epsilon^{\sigma\lambda\rho}\nabla_{\lambda}\mathcal{V}^{(0)}_{\rho} - m\mathcal{V}^{\sigma}_{(1)} = 0, \tag{31}$$

$$\frac{1}{2}\nabla^{\sigma}\mathcal{F}_{(0)} + \epsilon^{\sigma\lambda\rho}p_{\lambda}\mathcal{V}^{(1)}_{\rho} = 0. \tag{32}$$

Given Eqs. (24), (25), (29), and (31), it can be verified that Eq. (32) holds automatically, provided Eq. (30) is satisfied. So we see that Eq. (30) is the basic equation for the zeroth order solution of $\mathcal{V}_{(0)}$ and $\mathcal{F}_{(0)}$ which provides constraints for μ, T, u^{σ}. Inserting Eqs. (24) and (26) into Eq. (30), we obtain

$$\begin{aligned} 0 &= \nabla_{\sigma}\mathcal{V}^{\sigma}_{(0)} \\ &= \delta(p^2 - m^2)V'_{u\cdot p} \times \Big\{\bar{p}^2\Big[-Tu^{\sigma}\partial_{\sigma}\beta + \frac{1}{2}(\partial\cdot u)\Big] \\ &\quad - T\bar{p}^{\sigma}[\partial_{\sigma}(\beta\mu) + \beta QE_{\sigma}] + \Big(\bar{p}^{\alpha}\bar{p}^{\lambda} - \frac{1}{2}\bar{p}^2\Delta^{\alpha\lambda}\Big)\partial_{\langle\alpha}u_{\lambda\rangle} \\ &\quad + m^2Tu^{\sigma}\partial_{\sigma}\beta - (u\cdot p)Tu^{\sigma}\partial_{\sigma}(\beta\mu) \\ &\quad + (u\cdot p)\bar{p}^{\sigma}[T\partial_{\sigma}\beta + \dot{u}_{\sigma}]\Big\}. \end{aligned} \tag{33}$$

Here $V'_{u\cdot p}$ denotes the derivative of V with respect to $(u\cdot p)$ which does not include the derivative of the θ function, so we have

$$V'_{u\cdot p} \equiv \frac{\partial V}{\partial(u\cdot p)}\Big|_{\bar{\theta}} = \frac{\beta}{u\cdot p}\frac{\partial V}{\partial\beta} = -\beta\frac{\partial V}{\partial(\beta\mu)}. \tag{34}$$

We used the following notations:

$$\begin{aligned} &\beta = 1/T, \qquad \bar{p}^{\sigma} = p^{\sigma} - u^{\sigma}(u\cdot p) = \Delta^{\sigma\rho}p_{\rho} \\ &\partial_{\langle\alpha}u_{\lambda\rangle} = \frac{1}{2}\Delta_{\alpha\beta}\Delta_{\lambda\rho}(\partial^{\beta}u^{\rho} + \partial^{\rho}u^{\beta} - \Delta^{\beta\rho}\partial\cdot u), \end{aligned} \tag{35}$$

where the angular brackets $\langle\rangle$ denote the traceless symmetrized tensors. We also define E^{μ} and B^{μ} as

$$E^{\mu} = F^{\mu\nu}u_{\nu}, \qquad B^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\alpha}F_{\nu\alpha} = (-B, -E^2, E^1). \tag{36}$$

Note that the magnetic field $B = \partial_1 A^2 - \partial_2 A^1$ in (2 + 1)D is a pseudoscalar under rotation and the spatial components of B^{μ} are the Hall electric fields.

In order for Eq. (33) to be satisfied for any p (this condition will be relaxed after including collisions), we see that the following conditions must be fulfilled:

$$\begin{aligned} \partial_{\langle\alpha}u_{\lambda\rangle} &= 0, & \partial_\sigma(\beta\mu) + \beta Q E_\sigma &= 0, \\ \dot{u}_\sigma - \partial_\sigma \ln T &= 0, & \dot{T} &= 0, \\ u^\sigma\partial_\sigma(\beta\mu) &= 0, & \partial\cdot u &= 0, \end{aligned} \tag{37}$$

where we used the notation $\dot{X} = u^\sigma\partial_\sigma X$. The conditions $\partial\cdot u = 0$, $\partial_{\langle\alpha}u_{\lambda\rangle} = 0$ and $\partial_\sigma(\beta\mu) + \beta Q E_\sigma = 0$ imply that we neglect the bulk viscous pressure, shear viscous tensor, and heat conducting flow, respectively, so the system is dissipationless.

Contracting Eq. (31) with p_σ and using Eq. (29), we obtain

$$\mathcal{F}_{(1)} = \frac{1}{2}Q\epsilon^{\rho\sigma\xi}p_\rho F_{\sigma\xi}V\delta'(p^2-m^2) + \frac{1}{2}\hat{G}\delta(p^2-m^2), \tag{38}$$

where $\hat{G}$ is a function of x and p and will be determined later. From Eqs. (24), (31), and (38), we have

$$\begin{aligned} \mathcal{V}^\rho_{(1)} =& \frac{1}{m}p^\rho\mathcal{F}_{(1)} - \frac{1}{2m}\epsilon^{\rho\sigma\xi}\nabla_\sigma\mathcal{V}^{(0)}_\xi \\ =& -\frac{1}{2m}\delta(p^2-m^2)V'_{u\cdot p}\Big\{p^\rho\Big[\frac{1}{2}(u\cdot p)(u\cdot\omega) - (p\cdot\omega)\Big] \\ & -\frac{1}{2}(u\cdot\omega)m^2u^\rho + m^2\omega^\rho\Big\} + QmV\delta'(p^2-m^2)B^\rho \\ & + \hat{G}\frac{1}{2m}p^\rho\delta(p^2-m^2) + Q\frac{1}{2m}\bar{B}^\rho(u\cdot p)C\delta(p^2-m^2), \end{aligned} \tag{39}$$

where C comes from the $(u\cdot p)$ derivative acting on the θ functions in V of Eq. (26)

$$C = \frac{1}{2\pi^2}\sum_{e=\pm}\frac{e}{e^{\beta e(u\cdot p-\mu)}+1}\delta(eu\cdot p - |m|), \tag{40}$$

where we have used the definition of the vorticity $\omega^\rho = \epsilon^{\rho\sigma\xi}\partial_\sigma u_\xi$. We see that the vorticity and magnetic field emerge automatically in $\mathcal{V}^\mu$. They will contribute to the current and energy-momentum tensor.

For simplicity, we can rewrite the unknown function $\hat{G}$ in Eqs. (38) and (39) as

$$\hat{G} = G + V'_{u\cdot p}\Big[\frac{1}{2}(u\cdot p)(u\cdot\omega) - (p\cdot\omega)\Big] + QC(u\cdot B). \tag{41}$$

Then $\mathcal{V}^\mu_{(1)}$ and $\mathcal{F}_{(1)}$ become

$$\begin{aligned} \mathcal{V}^\rho_{(1)} =& -\frac{m}{2}\delta(p^2-m^2)V'_{u\cdot p}\Big[-\frac{1}{2}(u\cdot\omega)u^\rho + \omega^\rho\Big] \\ & + QmV\delta'(p^2-m^2)B^\rho \\ & + Q\frac{1}{2m}B^\rho(u\cdot p)C\delta(p^2-m^2) \\ & + G\frac{1}{2m}p^\rho\delta(p^2-m^2), \end{aligned} \tag{42}$$

$$\begin{aligned} \mathcal{F}_{(1)} =& Q(p\cdot B)V\delta'(p^2-m^2) + Q\frac{1}{2}(u\cdot B)C\delta(p^2-m^2) \\ & -\frac{1}{2}\delta(p^2-m^2)V'_{u\cdot p}\Big[-\frac{1}{2}(u\cdot p)(u\cdot\omega) + (p\cdot\omega)\Big] \\ & + \frac{G}{2}\delta(p^2-m^2), \end{aligned} \tag{43}$$

where we have used the fact that the combination of C and $\delta(p^2-m^2)$ leads to $\bar{p}^\sigma = 0$, since $C \propto \delta(eu\cdot p - |m|)$.

C. Second order

In order to determine G, we have to consider the second order constraints for $\mathcal{V}^\mu_{(1)}$. At the second order, Eqs. (15)–(18) become

$$p\cdot\mathcal{V}_{(2)} - m\mathcal{F}_{(2)} = 0, \tag{44}$$

$$\nabla\cdot\mathcal{V}_{(1)} = 0, \tag{45}$$

$$p^\rho\mathcal{F}_{(2)} - \frac{1}{2}\epsilon^{\rho\sigma\xi}\nabla_\sigma\mathcal{V}^{(1)}_\xi - m\mathcal{V}^\rho_{(2)} = 0, \tag{46}$$

$$\frac{1}{2}\nabla^\rho\mathcal{F}_{(1)} + \epsilon^{\rho\sigma\xi}p_\sigma\mathcal{V}^{(2)}_\xi = 0. \tag{47}$$

The first constraint is provided by Eq. (45). Substituting Eq. (46) into Eq. (47), we get

$$\begin{aligned} m\nabla^\rho\mathcal{F}_{(1)} &= \epsilon^{\rho\sigma\xi}p_\sigma\epsilon_{\xi\alpha\beta}\nabla^\alpha\mathcal{V}^\beta_{(1)} \\ &= p_\sigma[\nabla^\rho\mathcal{V}^\sigma_{(1)} - \nabla^\sigma\mathcal{V}^\rho_{(1)}], \end{aligned} \tag{48}$$

which gives the second constraint.

Substituting the solutions (42) and (43) into constraint Eqs. (45) and (48) and using identities in Appendix A, we arrive at

$$\nabla_\mu[p^\mu G\delta(p^2-m^2)] = 0. \tag{49}$$

We can show that $G = 0$ is a solution under certain physical constraints (see Appendix B), so we obtain

$$\begin{aligned} \mathcal{V}^\rho_{(1)} =& -\frac{m}{2}V'_{u\cdot p}\Big[-\frac{1}{2}(u\cdot\omega)u^\rho + \omega^\rho\Big]\delta(p^2-m^2) \\ & + QmV\delta'(p^2-m^2)B^\rho \\ & + \frac{1}{2m}QB^\rho(u\cdot p)C\delta(p^2-m^2). \end{aligned} \tag{50}$$

V. CURRENTS AND CONSERVATION LAWS

The medium part of the fermion number current can be obtained by integrating over p as in Eq. (12), combining with the vacuum part; the total current is then

$$j^{\sigma} = \left[n + \frac{1}{2}\xi(u\cdot\omega) + \xi_B(u\cdot B)\right]u^{\sigma} + \xi\bar{\omega}^{\sigma} + \xi_B\bar{B}^{\sigma}, \tag{51}$$

where the fermion number density n is given by Eq. (27) and two coefficients ξ and ξ_B are given by

$$\xi = mc_-(m), \qquad \xi_B = Q\mathrm{sign}(m)\left[c_+(m) - \frac{1}{4\pi}\right], \tag{52}$$

and $c_{\pm}(m)$ is defined by

$$c_{\pm}(m) = \frac{1}{4\pi}\left[\frac{1}{e^{\beta(|m|-\mu)}+1} \pm \frac{1}{e^{\beta(|m|+\mu)}+1}\right]. \tag{53}$$

First we look at the magnetic field part of the current. At the zero temperature limit $T\to 0$, we have

$$\xi_B = -\frac{Q}{4\pi}\,\mathrm{sign}(m)\theta(m^2-\mu^2), \tag{54}$$

which is consistent with the results of Refs. [50,51]. Now we look at the vorticity part of the current, in the $T\to 0$ limit, and we have

$$\xi = \frac{1}{4\pi}\,\mathrm{sign}(\mu)m\theta(\mu^2-m^2). \tag{55}$$

Note that there is no vacuum contribution to ξ due to the symmetry property of j^{σ}. The reason is as follows. The vacuum contribution, if there is any, must be in the form of $C'm$, where C' is a constant independent of μ, T, and Q, because under parity transformation m and ξ transform as $m\to -m$ and $\xi\to -\xi$. The current j^{σ} is also odd under charge conjugation transformation $\mu\to-\mu$ and $Q\to -Q$. However, the vacuum contribution does not change sign under charge conjugation transformation; hence, it is not allowed. Using identities in Appendix A and Eq. (37), one can verify the fermion number conservation

$$\partial_{\sigma}j^{\sigma} = 0. \tag{56}$$

This is different from the case in (3 + 1)D where the conservation is broken by anomaly.

By using Eq. (14), the energy-momentum tensor can be evaluated as

$$T^{\rho\sigma} = u^{\rho}u^{\sigma}[\varepsilon + \kappa(u\cdot\omega) + 2\kappa_B(u\cdot B)] - \Delta^{\rho\sigma}P + \kappa(u^{\rho}\bar{\omega}^{\sigma} + u^{\sigma}\bar{\omega}^{\rho}) + \kappa_B(u^{\rho}\bar{B}^{\sigma} + u^{\sigma}\bar{B}^{\rho}), \tag{57}$$

where the energy density ε and the pressure P are given by Eq. (28), and the coefficients κ and κ_B are given by

$$\begin{aligned}\kappa &= \frac{1}{8\pi}m\beta\int_{|m|}^{\infty}dE_pE_p\left[\frac{e^{\beta(E_p-\mu)}}{(e^{\beta(E_p-\mu)}+1)^2} + \frac{e^{\beta(E_p+\mu)}}{(e^{\beta(E_p+\mu)}+1)^2}\right]\\ &= -\frac{1}{4m}\beta\frac{\partial}{\partial\beta}(\varepsilon - 2P),\\ \kappa_B &= \frac{1}{2}mQc_-(m).\end{aligned} \tag{58}$$

Using identities in Appendix A, Eq. (37), $\beta\partial P/\partial\beta = -(\varepsilon + P)$ and $\beta\partial P/\partial(\beta\mu) = n$, we can verify the energy-momentum conservation

$$\partial_{\mu}T^{\mu\nu} = QF^{\nu\lambda}j_{\lambda}. \tag{59}$$

The entropy current [72] is defined as

$$\begin{aligned}s^{\sigma} &= \beta(Pu^{\sigma} - \mu j^{\sigma} + u_{\lambda}T^{\lambda\sigma})\\ &= su^{\sigma} + \beta\left[\left(\kappa - \frac{1}{2}\mu\xi\right)(u\cdot\omega) + (2\kappa_B - \mu\xi_B)(u\cdot B)\right]u^{\sigma}\\ &\quad + \beta(\kappa - \mu\xi)\bar{\omega}^{\sigma} + \beta(\kappa_B - \mu\xi_B)\bar{B}^{\sigma},\end{aligned} \tag{60}$$

where we have used the Gibbs-Duhem relation $sT = P + \varepsilon - \mu n$. Similarly we can confirm the entropy conservation,

$$\partial_{\sigma}s^{\sigma} = 0, \tag{61}$$

which means the system is nondissipative.

VI. LANDAU FRAME

The current and energy-momentum tensor Eqs. (51) and (57) are in a general frame. We can transform them to the Landau frame. To this end, we notice that the densities of effective energy, fermion number, and entropy are

$$\begin{aligned}\varepsilon_E &= u_{\rho}u_{\sigma}T^{\rho\sigma} = \varepsilon + \kappa(u\cdot\omega) + 2\kappa_B(u\cdot B),\\ n_E &= u_{\rho}j^{\rho} = n + \frac{1}{2}\xi(u\cdot\omega) + \xi_B(u\cdot B),\\ s_E &= u_{\rho}s^{\rho} = P + \varepsilon_E - n_E\mu.\end{aligned} \tag{62}$$

This is different from Israel-Stewart theory [72] where the energy and number density are corrected up to the second order when changing the frame. However, as we mentioned in the introduction, the number density is modified by the magnetic field and this is one of the properties in (2 + 1)D QED. So it is expected that the energy density is also modified by the magnetic field.

The Landau frame is defined as $\Delta_{\mu\nu}T^{\nu\alpha}u_{\alpha} = 0$, which defines a new fluid velocity

$$u_E^{\mu} = u^{\mu} + \frac{\kappa}{\epsilon + P}\bar{\omega}^{\mu} + \frac{\kappa_B}{\epsilon + P}\bar{B}^{\mu}.$$

Then $T^{\rho\sigma}$ is written as

$$\begin{aligned} T^{\rho\sigma} &= u_E^\rho u_E^\sigma(\varepsilon_E + P) - g^{\rho\sigma}P,\\ j^\sigma &= n_E u_E^\sigma + \xi^E\bar{\omega}^\sigma + \xi_B^E\bar{B}^\sigma,\\ s^\sigma &= s_E u_E^\sigma + \xi_s^E\bar{\omega}^\sigma + \xi_{sB}^E\bar{B}^\sigma, \end{aligned} \tag{63}$$

with the new coefficients

$$\begin{aligned} \xi^E &= \xi - \frac{n}{\epsilon + P}\kappa, \qquad \xi_B^E = \xi_B - \frac{n}{\epsilon + P}\kappa_B,\\ \xi_s^E &= \beta(\kappa - \mu\xi) - \frac{s}{\epsilon + P}\kappa,\\ \xi_{sB}^E &= \beta(\kappa_B - \mu\xi_B) - \frac{s}{\epsilon + P}\kappa_B. \end{aligned} \tag{64}$$

It is interesting to compare our results Eqs. (51) and (57) with the entropy principle analysis of Ref. [64]. We note that the $\tilde{\chi}_T$ and $\tilde{\chi}_E$ terms in Ref. [64] are actually our ξ and ξ_B terms, respectively, since $\tilde{E}_\sigma = -\bar{B}_\sigma$ and $\epsilon_{\eta\lambda\xi}u^\lambda\partial^\xi T = T\bar{\omega}_\eta$ with identities in Appendix A and constraints (37). Note that we do not have shear and Hall viscosity terms (η and $\tilde{\eta}$ terms), which are dissipative since we have no collision or dissipation as demonstrated by the entropy conservation in Eq. (61). Another dissipative term, the electric conductivity term (σ term) in Ref. [64], is also absent in our approximation.

VII. SUMMARY AND CONCLUSION

We derive the parity violating fluid dynamics of a fermionic system in (2 + 1) dimensions in quantum kinetic theory with the Winger function. Using a perturbative method in powers of the space-time derivative and electromagnetic field, we determine the Wigner function to the first order by solving a system of equations for the Wigner function to the second order. Our main results are Eqs. (51), (57), and (60). In the zeroth order, the Wigner function gives rise to the fermion number current, the entropy current, and the energy-momentum tensor of an ideal gas. In order for the first order equations to be satisfied, the constraints on the thermal variables are imposed. We then solve the Winger function up to the first order constrained by the second order equations. Integrating over the energy-momentum for the Wigner function, one obtains the fermion number current, the entropy current, and the energy-momentum tensor, where vorticity as well as electromagnetic field terms appear naturally. At zero temperature, the Hall conductivity is consistent with the previous result from quantum field theory. We also prove the conservation of entropy which indicates that the system is dissipationless.

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APPENDIX A: USEFUL IDENTITIES

In this appendix we give identities involving u_α, $\partial_\alpha u_\beta$, ω_α, B_α, and E_α under the conditions of Eq. (37). These identities are useful to verify Eq. (45).

First we list main identities concerning u_α, $\partial_\alpha u_\beta$, and ω_α. From $\dot{u}^\alpha = \partial^\alpha \ln T$ and $\partial_\sigma\omega^\sigma = 0$, we obtain

$$u_\beta\partial^\beta\omega_\rho = \omega_\beta\partial^\beta u_\rho. \tag{A1}$$

Then we have

$$u\cdot\partial(u\cdot\omega) = \dot{u}^\sigma\omega_\sigma = 0. \tag{A2}$$

We can derive

$$\begin{aligned} \partial_\rho u_\sigma &= \frac{1}{2}(\epsilon_{\rho\sigma\tau}\omega^\tau + u_\sigma\dot{u}_\rho + u_\rho\dot{u}_\sigma)\\ &= \frac{1}{2}(u\cdot\omega)\epsilon_{\rho\sigma\tau}u^\tau + u_\rho\dot{u}_\sigma,\\ \partial_\sigma\omega^\lambda &= \omega^\lambda\dot{u}_\sigma + \frac{1}{2}(u\cdot\omega)u^\lambda\dot{u}_\sigma + \frac{1}{2}(u\cdot\omega)\partial_\sigma u^\lambda, \end{aligned} \tag{A3}$$

where we have used $\partial_{\langle\alpha}u_{\lambda\rangle} = 0$ in Eq. (37) and (A1). Using Eq. (A3) we have

$$\begin{aligned} \omega^\rho\partial_\rho u_\sigma &= \frac{1}{2}(u\cdot\omega)\dot{u}_\sigma,\\ \omega^\sigma\partial_\rho u_\sigma &= \frac{1}{2}(u\cdot\omega)\dot{u}_\rho,\\ u_\sigma\partial_\rho\omega^\sigma &= \frac{3}{2}(u\cdot\omega)\dot{u}_\sigma. \end{aligned} \tag{A4}$$

Then we can derive identities for B_α and E_α. From $E^\rho = \epsilon^{\rho\alpha\beta}u_\alpha B_\beta$ we can easily see $E\cdot B = 0$. Using $\partial_\sigma(\beta\mu) = \beta Q E_\sigma$, $E^\rho = \epsilon^{\rho\alpha\beta}u_\alpha B_\beta$, and $\dot{u}^\alpha = \partial^\alpha \ln T$, we can derive

$$(E\cdot\omega)u^\lambda = B\cdot\partial u^\lambda, \tag{A5}$$

which leads to

$$E\cdot\omega = B\cdot\partial u^\lambda = \dot{u}_\beta B^\beta = u_\rho\partial^\rho(u\cdot B) = 0. \tag{A6}$$

APPENDIX B: PROOF OF $G = 0$

Let us fix G in $\mathcal{V}_1^\mu$ under some physical constraints. As we have shown in Eq. (7), the Wigner operator has the medium and vacuum parts. We obtain the medium part involving $\mathcal{V}_1^\mu$ by solving Eq. (9). If we take $V \to 0$, i.e., there are no particles in the system, the medium part should vanish. Therefore, G must be a function of V, its derivative $V'_{u\cdot p}$ and C. Simply, we can express G as polynomials of V, $V'_{u\cdot p}$, and C.

In the framework of the Boltzmann equation, the distribution function f can be expanded near equilibrium,

$$f = f_0 + f_1 + \cdots, \tag{B1}$$

where f_0 is the distribution function in equilibrium and the first order correction, $f_1 = -(\text{dissipative terms}) \times T\frac{\partial}{\partial(u\cdot p)} f_0$, is linear in f_0. In our case, V, $V'_{u\cdot p}$, and C (derivatives of V) correspond to f_0 and $T\frac{\partial}{\partial(u\cdot p)} f_0$. Therefore, we can assume G is also a linear combination of V, $V'_{u\cdot p}$, and C.

Including all possible contractions of the vectors u^μ, ω^μ, B^μ, and p^μ of the first order, we then have the following form for G:

$$\begin{aligned} G = V\Big[&\sum_{i=0}\frac{1}{m^{i+2}}(p\cdot\omega)(u\cdot p)^i X_{1,i} + \sum_{i=0}\frac{1}{m^{i+1}}(u\cdot\omega)(u\cdot p)^i X_{2,i} + \sum_{i=0}\frac{1}{m^{i+3}}(p\cdot B)(u\cdot p)^i X_{3,i} + \sum_{i=0}\frac{1}{m^{i+2}}(u\cdot B)(u\cdot p)^i X_{4,i}\Big] \\ + V'_{u\cdot p}\Big[&\sum_{i=0}\frac{1}{m^{i+1}}(p\cdot\omega)(u\cdot p)^i Y_{1,i} + \sum_{i=0}\frac{1}{m^{i}}(u\cdot\omega)(u\cdot p)^i Y_{2,i} + \sum_{i=0}\frac{1}{m^{i+2}}(p\cdot B)(u\cdot p)^i Y_{3,i} + \sum_{i=0}\frac{1}{m^{i+1}}(u\cdot B)(u\cdot p)^i Y_{4,i}\Big] \\ + C\Big[&\sum_{i=0}\frac{1}{m^{i+1}}(p\cdot\omega)(u\cdot p)^i Z_{1,i} + \sum_{i=0}\frac{1}{m^{i}}(u\cdot\omega)(u\cdot p)^i Z_{2,i} + \sum_{i=0}\frac{1}{m^{i+2}}(p\cdot B)(u\cdot p)^i Z_{3,i} + \sum_{i=0}\frac{1}{m^{i+1}}(u\cdot B)(u\cdot p)^i Z_{4,i}\Big], \end{aligned} \tag{B2}$$

where $X_{j,i}$, $Y_{j,i}$, $Z_{j,i}$ are dimensionless constants and all dependence on μ and T are through V, $V'_{u\cdot p}$, and C. From Eqs. (42) and (43), we neglect other complicated expressions, e.g., $\log(u\cdot p/m)$. We also assume that macroscopic quantities should not appear in denominators; e.g., terms like $1/(u\cdot B)$ will be divergent at vanishing magnetic field and should be absent in our discussions.

Since $C\delta(p^2-m^2) \propto \delta(u\cdot p \pm m)\delta(\mathbf{p})$, the $Z_{1,i}$ and $Z_{3,i}$ terms vanish, and only $Z_{2,0}$ and $Z_{4,0}$ terms survive. Although we consider a system of massive fermions, we would not expect any divergences in the current when we take the massless limit; i.e., $\int d^3p p^\mu G\delta(p^2-m^2)$ should be finite. Then all $1/m^i$ terms with $i>0$ should be gone. Then G becomes

$$G = C(u\cdot\omega)Z_{2,0} + C\frac{1}{m}(u\cdot B)Z_{4,0}. \tag{B3}$$

We have already proved that all terms in $\mathcal{V}_1^\mu$ except G satisfy the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda$. Therefore, G has to satisfy it separately, but it does not do so automatically unless $Z_{2,0}$ and $Z_{4,0}$ vanish. Finally we reach $G=0$.

APPENDIX C: ON THE CHOICE OF V IN EQ. (26)

In this appendix, we will derive Eqs. (27) and (28). We will also show that using $\theta(eu\cdot p)$ instead of $\theta(eu\cdot p - |m|)$ in Eq. (26) will not change the final result.

Integrating $\mathcal{V}_{(0)}^\sigma$ over (2 + 1)D momenta, we obtain the current at the leading order,

$$\begin{aligned} j^\sigma_{m(0)} &= \int d^3p p^\sigma V\delta(p^2-m^2) \\ &= \frac{1}{(2\pi)^2}u^\sigma \int dp_0 d^2p p_0\Big[\frac{1}{e^{(p_0-\mu)/T}+1}\theta(p_0-|m|) + \frac{1}{e^{(-p_0+\mu)/T}+1}\theta(-p_0-|m|)\Big]\frac{1}{E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)] \\ &= \frac{1}{(2\pi)^2}u^\sigma \int d^2p\Big[\frac{1}{e^{(E_p-\mu)/T}+1}\theta(E_p-|m|) - \frac{1}{e^{(E_p+\mu)/T}+1}\theta(E_p-|m|)\Big] \\ &= u^\sigma\frac{1}{2\pi}\int_{|m|}^\infty dE_p E_p\Big[\frac{1}{e^{\beta(E_p-\mu)}+1} - \frac{1}{e^{\beta(E_p+\mu)}+1}\Big], \end{aligned} \tag{C1}$$

where we have used $E_p dE_p = |\mathbf{p}|d|\mathbf{p}|$ and $p_0 \equiv u\cdot p$. The current in Eq. (C1) is just nu^σ in Eq. (27). We see the minus sign for the contribution from antifermions, which comes from $p_0\delta(p_0+E_p)$.

The energy-momentum tensor at the leading order can also be obtained by Eq. (14), which gives the energy density ε and pressure P in Eq. (28). The derivation is

$$\begin{aligned}T^{\sigma\rho}_{m(0)} &= \int d^3p\,p^\sigma p^\rho V\delta(p^2-m^2)\\ &= \int d^3p[\bar{p}^\sigma + (u\cdot p)u^\sigma][\bar{p}^\rho + (u\cdot p)u^\rho]V\delta(p^2-m^2)\\ &= \int d^3p\,\bar{p}^\sigma \bar{p}^\rho V\delta(p^2-m^2) + u^\sigma u^\rho \int d^3p(u\cdot p)^2 V\delta(p^2-m^2)\\ &= \frac{1}{2}\Delta^{\sigma\rho}\int d^3p\,\bar{p}^2 V\delta(p^2-m^2) + u^\sigma u^\rho \int d^3p(u\cdot p)^2 V\delta(p^2-m^2)\\ &= -\frac{1}{4\pi}\Delta^{\sigma\rho}\int_{|m|}^{\infty} dE_p(E_p^2-m^2)\left[\frac{1}{e^{\beta(E_p-\mu)}+1}+\frac{1}{e^{\beta(E_p+\mu)}+1}\right] + \frac{1}{2\pi}u^\sigma u^\rho\int_{|m|}^{\infty} dE_p E_p^2\left[\frac{1}{e^{\beta(E_p-\mu)}+1}+\frac{1}{e^{\beta(E_p+\mu)}+1}\right]\\ &= -\Delta^{\sigma\rho}P + u^\sigma u^\rho \varepsilon. \end{aligned} \tag{C2}$$

Note that $p_0^2 \equiv (u\cdot p)^2$ are involved in ε and P, so the antifermion contributions are always positive.

It makes no difference if we use $\theta(eu\cdot p)$ instead of $\theta(eu\cdot p - |m|)$ in V in Eq. (26). It can easily be seen in Eqs. (C1) and (C2) that the results of $j^\sigma_{m(0)}$ and $T^{\sigma\rho}_{m(0)}$ are the same, because

$$\theta(eu\cdot p)\delta(p^2-m^2) = \theta(eu\cdot p - |m|)\delta(p^2-m^2). \tag{C3}$$

Taking the $u\cdot p$ derivative in the θ function of V, C in Eq. (40) now becomes

$$C = \frac{1}{2\pi^2}\sum_{e=\pm}\frac{e}{e^{\beta e(u\cdot p-\mu)}+1}\delta(eu\cdot p), \tag{C4}$$

which gives $C\delta(p^2-m^2)=0$. So the C term in Eq. (50) vanishes. Note that the first term which is proportional to $V'_{u\cdot p}$ and defined in Eq. (34) gives the same result due to Eq. (C3).

Let us look at the second term in Eq. (50). Integrating over (2 + 1)D momentum, we obtain the correction to the Fermion number current at the first order (we suppress B^ρ),

$$\begin{aligned}&\frac{1}{2}Qm\int d^3p\frac{V}{p_0}\frac{\partial}{\partial p_0}\delta(p^2-m^2)\\ &\quad = -\frac{1}{2}Qm\int d^3p\left(\frac{1}{p_0}\frac{\partial V}{\partial p_0} - \frac{V}{p_0^2}\right)\delta(p^2-m^2)\\ &\quad = Q\,\mathrm{sign}(m)c_+(m).\end{aligned} \tag{C5}$$

Combining with the vacuum part (1), the total current is the same as Eq. (51). We have used the fact that the C part in $\partial V/\partial p_0$ combines with $\delta(p^2-m^2)$ to give zero. So using Eq. (C3) for the nonvanishing part we see that replacing $\theta(eu\cdot p - |m|)$ with $\theta(eu\cdot p)$ in V does not make any difference.

We can also evaluate the correction to the energy-momentum tensor at the first order by using Eq. (14). Similarly one can prove that the result is the same after making the replacement $\theta(eu\cdot p - |m|)\to\theta(eu\cdot p)$.

Therefore, we have demonstrated that the replacement $\theta(eu\cdot p - |m|)\to\theta(eu\cdot p)$ in V in Eq. (26) does not make any difference in the final result.

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