Berry Curvature and Four-Dimensional Monopoles in the Relativistic Chiral Kinetic Equation

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We derive a relativistic chiral kinetic equation with manifest Lorentz covariance from Wigner functions of spin-1/2 massless fermions in a constant background electromagnetic field. It contains vorticity terms and a four-dimensional Euclidean Berry monopole which gives an axial anomaly. By integrating out the zeroth component of the 4-momentum p, we reproduce the previous three-dimensional results derived from the Hamiltonian approach, together with the newly derived vorticity terms. The phase space continuity equation has an anomalous source term proportional to the product of electric and magnetic fields ($F_{\sigma\rho}\tilde{F}^{\sigma\rho} \sim E_{\sigma}B^{\sigma}$). This provides a unified interpretation of the chiral magnetic and vortical effects, chiral anomaly, Berry curvature, and the Berry monopole in the framework of Wigner functions.

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The Berry phase is a topological phase factor acquired by an eigenenergy state when it undergoes adiabatic evolution along a loop in parameter space [1]. It is in close analogy to the Aharonov-Bohm phase when a charged particle moves in a loop enclosing a magnetic flux, while the Berry curvature is like the magnetic field. The integral of the Berry curvature over a closed surface can be quantized as integers known as Chern-Simons numbers, which is similar to the Dirac magnetic monopole and has a deep connection with the quantum Hall effect. The Berry phase is a beautiful, simple, and universal structure in quantum physics and has many interesting applications; for a recent review of the Berry phase in condensed matter physics, see, e.g., Ref. [2].

Recently, it has been found that features of the Berry phase due to a three-dimensional momentum monopole emerge in a chiral kinetic equation without manifest Lorentz covariance [3,4]. A semiclassical kinetic equation has also been derived in an electron system with Berry curvature [5]. Chiral anomaly is an important quantum effect which is absent at the classical level. It is manifested in the chiral magnetic (CME) and vortical effect (CVE) [6–8] as electric currents induced by the magnetic field and vorticity. Such effects and related topics have been investigated within a variety of approaches, such as AdS/CFT correspondence [9–13], relativistic hydrodynamics [14–18], and quantum field theory [7,19–25].

In this Letter, we will derive a new chiral kinetic equation with manifest Lorentz covariance from the Wigner function [26]. Such an equation can provide a semiclassical description of quantum transport phenomena. We will show that such a chiral kinetic equation incorporates features of the Berry curvature and four-dimensional Euclidean monopole. These results reveal the inherent connection between the Berry phase and gauge invariant Wigner functions. One advantage of our approach is that the vorticity effect in the chiral kinetic equation can be derived straightforwardly which, apparently, is not the case in other approaches. We also show that the previous noncovariant kinetic equation [3,4] is equivalent to our covariant one in the case of zero vorticity after we integrate over the zeroth momentum p_0. The relativistic chiral kinetic equation in our approach is quite general and valid not only for Fermi liquid as in Refs. [3,4] but for any relativistic fermionic system. The phase space continuity or Liouville's theorem can be shown to be broken by an anomalous term proportional to the product of electric and magnetic fields. So the phase space measure is not conserved. It is modified by a factor related to the Berry curvature. We will also show that the conservation law of the right- and left-hand currents is broken by anomalous terms, which can be given by the flux of a four-dimensional monopole in Euclidean momentum space. Therefore we provide a unified interpretation of a variety of properties such as CME/CVE, chiral anomaly, Berry curvature, and four-dimensional Euclidean monopole in the framework of Wigner functions. We will use the metric convention $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Equation of motion with Berry curvature in three dimensions.—We will follow an example in Ref. [3] to illustrate the concept of Berry curvature. We consider a Hamiltonian $H' = \boldsymbol{\sigma} \cdot \mathbf{p}$ for spin-1/2 fermions in addition to the normal part $H(\mathbf{p}, \mathbf{x})$, where $\boldsymbol{\sigma}$ are Pauli matrices. Under an adiabatic evolution, the path-integral action for fermions with positive helicity is

$$S = \int dt[\dot{\mathbf{x}} \cdot \mathbf{p} + \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) - \dot{\mathbf{p}} \cdot \mathbf{a}(\mathbf{p}) - H(\mathbf{p}, \mathbf{x})], \quad (1)$$

where $\mathbf{A}(\mathbf{x})$ is the electromagnetic vector potential and $\mathbf{a}(\mathbf{p})$ is the vector potential in momentum space resulting from diagonalizing H' in the path integral. We can generalize the coordinate variables by combining $\mathbf{p}$ and $\mathbf{x}$, $\xi_a = (\mathbf{p}, \mathbf{x})$ with $a = 1, 2, \ldots, 6$. The action can be cast into a compact form:

$$S = \int dt[-\gamma_a(\xi)\dot{\xi}_a - H(\xi)], \quad (2)$$

where $\gamma_a = [\mathbf{a}(\mathbf{p}), -\mathbf{p} - \mathbf{A}(\mathbf{x})]$. The equations of motion are

$$\gamma_{ab}\dot{\xi}_b = -\frac{\partial H(\xi)}{\partial \xi_a}, \quad (3)$$

where $\gamma_{ab} \equiv \partial_a \gamma_b(\xi) - \partial_b \gamma_a(\xi)$ is given by

$$[\gamma_{ab}] = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 & -1 & 0 & 0 \\ -\Omega_3 & 0 & \Omega_1 & 0 & -1 & 0 \\ \Omega_2 & -\Omega_1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -B_3 & B_2 \\ 0 & 1 & 0 & B_3 & 0 & -B_1 \\ 0 & 0 & 1 & -B_2 & B_1 & 0 \end{bmatrix}, \quad (4)$$

where $\boldsymbol{\Omega} = \nabla_{\mathbf{p}} \times \mathbf{a}(\mathbf{p})$ is the Berry curvature and $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x})$ is the three-dimensional magnetic field. The determinant of $[\gamma_{ab}]$ is $\det[\gamma_{ab}] = (1 + \boldsymbol{\Omega} \cdot \mathbf{B})^2$. We see that the invariant phase space volume becomes $\sqrt{\det[\gamma_{ab}]}d^3\mathbf{x}d^3\mathbf{p}$, where $\sqrt{\det[\gamma_{ab}]} = |1 + \boldsymbol{\Omega} \cdot \mathbf{B}|$ indicates the change of phase space volume with time [27].

The Wigner function approach.—In an alternative quantum kinetic theory approach, the classical phase-space distribution $f(x, p)$ is replaced by the Wigner function $W(x, p)$ in space-time x and 4-momentum p, defined as the ensemble average of the Wigner operator [28–30] for spin-1/2 fermions:

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta(x_+) U(x_+, x_-) \psi_\alpha(x_-), \quad (5)$$

where ψ_α and $\bar{\psi}_\beta$ are Dirac spinor fields, $x_\pm \equiv x \pm (1/2)y$ are two space-time points centered at x with space-time separation y, and the gauge link U,

$$U(x_+, x_-) \equiv e^{-iQ\int_{x_-}^{x_+} dz^\mu A_\mu(z)}, \quad (6)$$

ensures the gauge invariance of $\hat{W}_{\alpha\beta}$. Here Q is the electromagnetic charge of the fermions, and A_μ is the electromagnetic vector potential. To simplify the quantum kinetic equation under a background field, we consider a massless and collisionless fermionic system in a constant external electromagnetic field $F_{\mu\nu}$ in the lab frame. The Wigner function for spin-1/2 fermions is a matrix in Dirac space and satisfies the quantum kinetic equation [28–30] $\gamma_\mu(p^\mu + (i/2)\nabla^\mu)W(x, p) = 0$, where γ^μ's are Dirac matrices and $\nabla^\mu \equiv \partial_x^\mu - QF_\nu^\mu \partial_p^\nu$. The Wigner function can be decomposed in terms of 16 independent generators of the Clifford algebra whose coefficients are scalar, pseudoscalar, vector, axial vector, and tensor, respectively. The vector $\mathcal{V}_\mu(x, p)$ and axial-vector $\mathcal{A}_\mu(x, p)$ component of the Wigner function can be determined by the quantum kinetic equations (i.e., Eqs. (5)–(8) of Ref. [26]) to the first order of space-time derivative ∂_x and the field strength $F_{\mu\nu}$:

$$\begin{aligned} \mathcal{Z}^\mu = {} & p^\mu \delta(p^2) Z_0 + \frac{1}{2} p_\nu [u^\mu \omega^\nu - u^\nu \omega^\mu] \frac{\partial \bar{Z}_0}{\partial(u \cdot p)} \delta(p^2) \\ & - Q p_\nu [u^\mu B^\nu - u^\nu B^\mu] \bar{Z}_0 \delta'(p^2) \\ & + Q \epsilon^{\mu\lambda\rho\sigma} u_\lambda p_\rho E_\sigma \bar{Z}_0 \delta'(p^2), \end{aligned} \quad (7)$$

where $\mathcal{Z} = (\mathcal{V}, \mathcal{A})$, $Z_0 = (V_0, A_0)$, and $\bar{Z}_0 = (A_0, V_0)$, with the first-order solutions V_0 and A_0 given by

$$[V_0, A_0] = \sum_{s=\pm 1} \theta(su \cdot p)[(f_{s,R} + f_{s,L}), (f_{s,R} - f_{s,L})],$$

$$f_{s,\chi} = \frac{2}{(2\pi)^3} \frac{1}{e^{s(u\cdot p - \mu_\chi)/T} + 1}, \quad (\chi = R, L), \quad (8)$$

where $R(L)$ denotes the right- (left-) handed fermions and $\mu_{R,L} = \mu \pm \mu_5$. We have used notations $E_\sigma = u^\rho F_{\sigma\rho}$, $B_\sigma = (1/2)\epsilon_{\sigma\mu\nu\rho} u^\mu F^{\nu\rho}$, and $\omega_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} u^\nu \partial^\rho u^\sigma$, which depend on x only via the fluid velocity $u(x)$. We use $\mathcal{Z}_0^\mu$ to denote the zeroth-order term $p^\mu \delta(p^2) Z_0$ in Eq. (7) and $\mathcal{Z}_1^\mu$ for the first-order terms.

The vector and axial-vector current and the energy-momentum tensor can be derived from $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ in Eq. (7) by integrating over momentum: $j^\mu = \int d^4p \mathcal{V}^\mu$, $j_5^\mu = \int d^4p \mathcal{A}^\mu$, and $T^{\mu\nu} = (1/2)\int d^4p (p^\mu \mathcal{V}^\nu + p^\nu \mathcal{V}^\mu)$. The current j^μ contains two parts proportional to the magnetic field and vorticity, known as the CME and CVE [6–8,14], respectively. So both effects are contained in the Wigner function [26]. These currents and $T^{\mu\nu}$ are shown to obey conservation equations [26]: $\partial_\mu j^\mu = 0$, $\partial_\mu j_5^\mu = -(Q^2/2\pi^2) E \cdot B$, and $\partial_\mu T^{\mu\nu} = QF^{\nu\rho} j_\rho$.

Lorentz covariant chiral kinetic equation.—Now we try to derive a new form of Lorentz covariant chiral kinetic equation in four dimensions from Eq. (6) of Ref. [26], i.e., $\nabla_\mu \mathcal{Z}^\mu = 0$, which holds for the zeroth- and first-order Wigner function $\mathcal{Z}_0^\mu$ and $\mathcal{Z}_1^\mu$ separately. The zeroth-order equation $\nabla_\mu \mathcal{Z}_0^\mu = 0$ can be rewritten as

$$\begin{aligned} \nabla_\mu \mathcal{Z}_0^\mu &= (\partial_\mu^x - QF_{\mu\nu}\partial_p^\nu)[p^\mu \delta(p^2) Z_0] \\ &= \delta(p^2)[p^\mu \partial_\mu^x - Q p^\mu F_{\mu\nu} \partial_p^\nu] Z_0 = 0. \end{aligned} \quad (9)$$

Here Z_0 is the phase space distribution function and given in Eq. (8). Equation (9) is a Vlasov-like equation, from which we can extract $dx^\sigma/d\tau = p^\sigma/m_0$ and $dp^\mu/d\tau = Qp^\mu F_{\mu\nu}/m_0$, where τ is a world-line parameter, m_0 is a quantity with mass dimension (but not the fermion mass since we are considering massless fermions) which can be scaled away by collision terms, and $Qp^\mu F_{\mu\nu}/m_0$ is a general Lorentz force.

Now we rewrite the first-order equation $\nabla_\mu \mathcal{Z}_1^\mu = 0$ in a way that only terms of spatial and momentum derivatives of $\bar{Z}_0$ are kept:

$$\begin{aligned}\nabla_\mu \mathcal{Z}_1^\mu &= Q\delta(p^2)[(u\cdot b)B^\mu - (b\cdot B)u^\mu + \epsilon^{\mu\nu\rho\sigma}u_\nu b_\rho E_\sigma] \\ &\times \partial_\mu \bar{Z}_0 + \delta(p^2)\left[\frac{1}{2}\omega^\mu + \omega^\mu (p\cdot u)(b\cdot u)\right. \\ &\left. - 2u^\mu (p\cdot\omega)(b\cdot u)\right]\partial_\mu \bar{Z}_0 \\ &- Q^2\delta(p^2)(E\cdot B)b^\sigma \partial_\sigma^p \bar{Z}_0 + Q\delta(p^2) \\ &\times\left[\frac{1}{2}(\omega\cdot E)u^\sigma + \frac{1}{p^2}(p\cdot\omega)p_\eta F^{\sigma\eta}\right]\partial_\sigma^p \bar{Z}_0 = 0,\end{aligned} \tag{10}$$

where $b^\sigma \equiv -p^\sigma/p^2$. The first two terms are proportional to $\partial_\mu \bar{Z}_0$, and the last two terms are proportional to $\partial_\sigma^p \bar{Z}_0$. We will show that $\delta(p^2)b^\sigma$ is a four-dimensional monopole in Euclidean momentum space. Combining Eqs. (9) and (10), we obtain the Lorentz covariant chiral kinetic equation

$$\begin{aligned}&\frac{1}{2}\nabla_\mu(\mathcal{V}^\mu \pm \mathcal{A}^\mu) = 0 \\ &\to \delta(p^2)\left[\frac{dx^\sigma}{d\tau}\partial_\sigma^x + \frac{dp^\sigma}{d\tau}\partial_\sigma^p\right]f_{R/L} = 0,\end{aligned} \tag{11}$$

where the upper (lower) sign corresponds to the right- (left-) hand distribution and $dx^\sigma/d\tau$ and $dp^\sigma/d\tau$ are given by

$$\begin{aligned}m_0\frac{dx^\sigma}{d\tau} &= p^\sigma \pm Q[(u\cdot b)B^\sigma - (b\cdot B)u^\sigma + \epsilon^{\sigma\alpha\beta\gamma}u_\alpha b_\beta E_\gamma] \\ &\pm\left[\frac{1}{2}\omega^\sigma + \omega^\sigma(p\cdot u)(b\cdot u) - 2u^\sigma(p\cdot\omega)(b\cdot u)\right], \\ m_0\frac{dp^\sigma}{d\tau} &= -Qp_\rho F^{\rho\sigma} \mp Q^2(E\cdot B)b^\sigma \pm Q\frac{1}{2}(\omega\cdot E)u^\sigma \\ &\mp Q(p\cdot\omega)b_\eta F^{\sigma\eta}.\end{aligned} \tag{12}$$

Here we have used notations

$$f_{R/L} \equiv \frac{1}{2}(V_0 \pm A_0) = \sum_{s=\pm1}\theta(su\cdot p)f_{s,R/L}. \tag{13}$$

Using the spatial and momentum divergences

$$\begin{aligned}&\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)\right] = 0, \\ &\partial_\sigma^p\left[\frac{dp^\sigma}{d\tau}\delta(p^2)\right] = \mp Q^2(E\cdot B)\partial_\sigma^p[b^\sigma\delta(p^2)],\end{aligned} \tag{14}$$

we obtain

$$\begin{aligned}&\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)\right]f_{R/L} + \partial_\sigma^p\left[\frac{dp^\sigma}{d\tau}\delta(p^2)\right]f_{R/L} \\ &= \mp Q^2(E\cdot B)\partial_\sigma^p[b^\sigma\delta(p^2)]f_{R/L}.\end{aligned} \tag{15}$$

It is interesting to see that $\delta(p^2)dx^\sigma/d\tau$ is conserved but $\delta(p^2)dp^\sigma/d\tau$ is not. We can combine Eq. (15) with the chiral kinetic equation (11) to obtain the Liouville equation or the phase space continuity equation

$$\begin{aligned}&\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)f_{R/L}\right] + \partial_\sigma^p\left[\frac{dp^\sigma}{d\tau}\delta(p^2)f_{R/L}\right] \\ &= \mp Q^2(E\cdot B)\partial_\sigma^p[b^\sigma\delta(p^2)]f_{R/L}.\end{aligned} \tag{16}$$

In deriving Eqs. (11)–(16), we have used the conditions $u^\mu\partial_\mu\omega^\nu = u^\mu\partial_\mu B^\nu = 0$, $\partial_\mu\omega^\mu = 0$, $\partial_\mu B^\mu = 2(\omega\cdot E)$, $\partial^\mu u^\nu = \epsilon^{\mu\nu\tau\lambda}u_\tau\omega_\lambda$, $\epsilon^{\sigma\rho\eta\xi}u_\rho\partial_\sigma E_\xi = 0$, $\epsilon_{\mu\nu\sigma\rho}u^\mu\omega^\nu B^\sigma = 0$, $\epsilon^{\sigma\rho\alpha\beta}\epsilon_{\sigma\rho\mu\nu} = -2\delta_\mu^\alpha\delta_\nu^\beta + 2\delta_\nu^\alpha\delta_\mu^\beta$, $\partial_\sigma^p b^\sigma = -2/p^2$, and $\partial_\sigma^p[\delta(p^2)] = 2b_\sigma\delta(p^2)$. We see in Eq. (16) the breaking of continuity for the phase space density by an anomalous term proportional to $E\cdot B$.

The vector currents for right- and left-hand fermions can be given by integration over 4-momentum from $dx^\sigma/d\tau$ as

$$j_{R/L}^\sigma = \int d^4p\,\delta(p^2)\frac{dx^\sigma}{d\tau}f_{R/L} = \frac{1}{2}(j^\sigma \pm j_5^\sigma), \tag{17}$$

where j^σ and j_5^σ are given in Eqs. (17)–(18) of Ref. [26]. The energy-momentum tensor can also be obtained from $dx^\sigma/d\tau$:

$$T^{\sigma\rho} = \int d^4p\left(p^\sigma\frac{dx^\rho}{d\tau} + p^\rho\frac{dx^\sigma}{d\tau}\right), \tag{18}$$

which gives Eq. (19) of Ref. [26]. Note that the vorticity terms in Eqs. (11) and (12) are necessary for the presence of the CVE in $j_{R/L}^\sigma$ and $T^{\sigma\rho}$.

Chiral kinetic equation in three dimensions.—We can obtain the chiral kinetic equation in three dimensions by integration over p_0 for the Lorentz covariant chiral kinetic equation (11) as

$$\int dp_0\,\delta(p^2)\left[\frac{dx^\sigma}{d\tau}\partial_\sigma^x f_{R/L} + \frac{dp^\sigma}{d\tau}\partial_\sigma^p f_{R/L}\right] = 0, \tag{19}$$

which amounts to calculating the following integrals:

$$I_n = \int dp_0\,\delta(p^2)\frac{p_0^n}{p^2}F(x,p),$$

with $n = 0, 1, 2$. We use the $i\epsilon$ prescription

$$\delta(x)\mathcal{P}\frac{1}{x} = -\frac{1}{2\pi}\,\mathrm{Im}\frac{1}{(x+i\epsilon)^2} \tag{20}$$

to evaluate the integrals by enclosing the pole $p_0 = |\mathbf{p}| - i\epsilon$ in the lower half-plane of p_0, where ϵ is a small positive number. Then we can derive the chiral kinetic equation in three dimensions from Eq. (19):

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0, \quad (21)$$

where $dt/d\tau$, $d\mathbf{x}/d\tau$, and $d\mathbf{p}/d\tau$ are given by

$$\frac{dt}{d\tau} = 1 \pm Q\mathbf{\Omega}\cdot\mathbf{B} \pm 4|\mathbf{p}|(\mathbf{\Omega}\cdot\boldsymbol{\omega}),$$
$$\frac{d\mathbf{x}}{d\tau} = \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\mathbf{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\mathbf{\Omega}) \pm \frac{1}{|\mathbf{p}|}\boldsymbol{\omega},$$
$$\frac{d\mathbf{p}}{d\tau} = Q(\mathbf{E}+\hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega} \mp Q|\mathbf{p}|(\mathbf{E}\cdot\boldsymbol{\omega})\mathbf{\Omega} \pm 3Q(\mathbf{\Omega}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\hat{\mathbf{p}}, \quad (22)$$

with $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ and the three-dimensional Berry curvature $\mathbf{\Omega} = \mathbf{p}/(2|\mathbf{p}|^3)$ [3,4]. Turning off the ω terms in Eq. (22), we reproduce Eqs. (14) and (15) in Ref. [4]. Note that the ω terms in the chiral kinetic equation in three dimensions come naturally from the Lorentz covariant chiral kinetic equation in four dimensions.

Anomaly and four-dimensional Berry monopole.—From Eq. (7), the anomalous conservation law of the left- and right-hand current can be derived from $\partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu = -(Q^2/2\pi^2)E\cdot B$:

$$\partial_\rho j^\rho_{R/L} = \mp\frac{Q^2}{4\pi^2}(E\cdot B). \quad (23)$$

On the other hand, we can understand the chiral anomaly in Eq. (23) from the perspective of a four-dimensional Berry monopole. To this end, we act ∂_σ on Eq. (17), use Eq. (16), and carry out the integral in Euclidean space:

$$\begin{aligned}\partial_\sigma j^\sigma_{R/L} &= \mp Q^2(E\cdot B)\int d^4p\,\partial^p_\sigma[b^\sigma\delta(p^2)]f_{R/L}\\ &= \mp Q^2(E\cdot B)\frac{1}{\pi}\,\mathrm{Im}\int_{-i\infty}^{i\infty}dp_0d^3p\\ &\quad\times\partial^p_\sigma\left[\frac{p^\sigma}{p^2}\frac{1}{p^2+i\epsilon}\right]f_{R/L}\\ &= \mp Q^2(E\cdot B)\frac{1}{\pi}\int_{-\infty}^{\infty}dp_4d^3p\,\partial^{p_E}_\sigma\left[\frac{p^\sigma_E}{p^4_E}\right]f_{R/L}\\ &= \mp\frac{Q^2}{4\pi^2}(E\cdot B). \end{aligned}\quad (24)$$

We have used $\pi\delta(x) = -\mathrm{Im}[1/(x+i\epsilon)]$ and taken analytic continuation $p_4 = ip_0$ and $p^2 = -p_E^2$. Note that in the second equality of Eq. (24) the poles in Minkowski space are $p_0 = \pm\sqrt{|\mathbf{p}|^2 - i\epsilon} = \pm|\mathbf{p}| \mp i\epsilon$; in order to avoid these poles in Wick rotation, the integral limit of p_0 should be $[-i\infty, i\infty]$, which corresponds to $[\infty, -\infty]$ for the p_4 integral. We have also used

$$\partial^{p_E}_\sigma(p^\sigma_E/p^4_E) = 2\pi^2\delta^{(4)}(p^\sigma_E). \quad (25)$$

Although for $p_E \neq 0$, we have $\partial^{p_E}_\sigma(p^\sigma_E/p^4_E) = 0$, the integral is nonvanishing, since

$$\int d^4p_E\partial^{p_E}_\sigma(p^\sigma_E/p^4_E) = \oint dS_{3,\sigma}p^\sigma_E/p^4_E = 2\pi^2. \quad (26)$$

Note that the n-dimensional volume of the n-dimensional sphere or the hypersurface area of the $(n+1)$-dimensional ball with radius R is given by $S_n = \{2\pi^{(n+1)/2}/\Gamma[(n+1)/2]\}R^n$. So we see that $\delta(p^2)b^\sigma$ plays the role of the Berry curvature of a four-dimensional monopole in Euclidean momentum space, where the singular point of the monopole is located at $p_E = 0$. This is related to the three-dimensional case by simply imposing the on-shell condition $\int dp_0\delta(p^2)b^\sigma = (0, \mathbf{\Omega}/2)$.

In fact, Eq. (5) has encoded the Berry phase already. When a plus helicity fermion moves under a weak external electromagnetic field, its momentum changes adiabatically from k to k'. The Wigner function could develop a nontrivial Berry phase related to the matrix element $\langle k+\delta k|\mathrm{tr}\hat{W}\gamma^0|k\rangle \propto u^\dagger(k')u(k) \simeq e^{i\delta k\cdot a}$, where we have $u^\dagger(k)u(k) = 1$ and $a^\alpha \equiv iu^\dagger(k)\partial^\alpha_k u(k) = (a^0, \mathbf{a})$. The curvature $\varpi^{\alpha\beta} \equiv \partial^\alpha_k a^\beta - \partial^\beta_k a^\alpha$ yields a Berry magnetic field $\varpi^{ij} = \epsilon^{ijk}\Omega^k$ but no Berry electric field ($\varpi^{0i} = 0$). Thus, the three-dimensional $\mathbf{a}$ and $\mathbf{\Omega}$ are naturally embedded in the four-dimensional result.

In summary, we have shown that the Berry curvature and a four-dimensional monopole in Euclidean momentum space emerge in a new chiral kinetic equation with manifest Lorentz covariance. The chiral anomaly can be interpreted as the flux of this four-dimensional monopole. There are vorticity terms in this chiral kinetic equation which are necessary for the presence of the chiral vortical effect. The three-dimensional chiral kinetic equation can be obtained from the Lorentz covariant one by integration over the zeroth component of the 4-momentum. It contains vorticity terms in addition to what was previously derived in the Hamiltonian approach. The phase space continuity equation has an anomalous source term proportional to the product of electric and magnetic fields. Our approach to the chiral kinetic equation is quite general and valid for relativistic fermionic systems.

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Note added.—Recently, we learned that Son and Yamamoto were also working on a similar topic [31].

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