Natural emergence of cosmological constant and dark radiation from the Stephenson-Kilmister-Yang-Camenzind theory of gravity

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We show that the Stephenson-Kilmister-Yang (SKY) equation combined with Camenzind's matter current term naturally provides the cosmological constant and dark radiation as integration constants of the Stephenson-Kilmister-Yang-Camenzind (SKYC) field equation. To characterize the property of the dark radiation, we develop a method to separate it from the ordinary radiation. We find a special property of Camenzind's matter current, namely that the solution space for radiation in fact belongs to that of the vacuum solution of the SKY equation. We also find that this matter current does not obey the conservation condition suggested by Kilmister. Finally, we discuss the possible role of dark radiation emergent from the SKYC theory in recent cosmic microwave background observations and its implications to the inflation scenario.

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I. INTRODUCTION

Various attempts have been made throughout the last century to unify all fundamental interactions. One pioneering effort was made by Weyl in 1918 [1,2], where he assumed that all physical laws should be invariant under conformal transformation. This seminal viewpoint introduced by Weyl is now known as the gauge invariance. Based on the principle of gauge invariance, Weyl reformulated the connection in Riemannian geometry to unify gravity and electromagnetism. However, this theory is problematic because of the path-dependent nature of the observer's clock, as pointed out by Einstein [3].

The gauge theory of Yang and Mills has inspired a new type of relationship between geometry and physics [4]. Such a theory can be considered as a vector bundle on a Riemannian manifold [5–7], where a section of a vector bundle corresponds to a matter field and a connection on the Riemannian manifold corresponds to a gauge field. A gauge theory can therefore be recognized as a functional (called the Yang-Mills functional) that acts on a metric connection on a vector bundle. The Yang-Mills functional is also invariant under a gauge transformation. Thus, Yang-Mills theory exhibits a close correspondence with the vector bundle theory in differential geometry.

We now know that all fundamental interactions except general relativity (GR)— that is, the electromagnetic, weak, and strong interactions—can be described in the language of gauge theory. The pioneering works that formulated Einstein's GR into a gauge theoretical framework started with Utiyama, who suggested that GR can be written in the language of gauge theory if the symmetry group is chosen either as the Poincaré group or the translational gauge group [8–11].

Stephenson, Kilmister, and Newman proposed a gravity theory analogous to the gauge theory [12,13], where the action is composed of quadratic curvature tensor terms without a linear one. The action gives rise to not only a higher-derivative equation that is consistent with Einstein's GR, but additional constraint equations as well. Later, Yang put forward another gauge theory of gravity that satisfies the $GL(n)$ symmetry group [14], and succeeded in deriving the same higher-derivative equation without the additional constraint equations. Following this convention, we shall refer to the higher-derivative field equation of gravity as the Stephenson-Kilmister-Yang (SKY) equation. The SKY equation reproduces all solutions of the vacuum Einstein equation [12–18]. Later, Camenzind proposed a matter current term for the SKY field equation [19], although it was not deduced from an action. This task was fulfilled by Cook in 2009 [20]. We shall call the complete theory that includes both pure space and matter contributions the Stephenson-Kilmister-Yang-Camenzind (SKYC) gravity.

A renormalizable quantum theory of gravity based on the Einstein-Hilbert action is known to be difficult to attain [21–23]. The extension of GR that includes quadratic curvature tensor terms, i.e., the so-called *higher-derivative gravity theory*, on the contrary, has been shown to be renormalizable [24–26] and asymptotically free [27,28]. This is mainly because issues such as renormalization are determined by the high-energy or ultraviolet behavior of

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the theory where the higher-derivative terms dominate, which converge faster than the linear Einstein-Hilbert term does. These investigations into the renormalization of higher-derivative gravity theories should be applicable to SKYC gravity, which is quadratic in curvature tensors.

While a higher-derivative theory of quantum gravity is better behaved in the ultraviolet limit and is renormalizable, it generally induces ghost excitations, which would render the theory pathological. Salam and Strathdee argued, based on the consideration of quantum running effects, that if the cutoff scale of the theory is smaller than the mass scale of the ghost mode, then the latter would be innocuous in such higher-derivative gravity theories [29]. Setting the cutoff at the Planck scale, a higher-derivative gravity theory may on the other hand be regarded as a provisional effective theory [30,31]. Antoniadis and Tomboulis showed, in addition, that the presence of such a massive spin-2 ghost in the bare propagator is inconclusive, since the excitation is unstable [32]. They further showed that such a ghost is gauge dependent, while the S-matrix of such a theory is unitary and gauge invariant. Therefore the ghost cannot contribute to the gauge-invariant absorptive part of the theory. We expect that, with similar considerations, the tensor ghost stemming from higher-order derivatives of the theory can be rendered harmless. These issues, however, are not the focus of this paper. We will be mainly concerned with the classical aspects of the higher-derivative, or higher-order, SKYC gravity theory.

The cosmological constant has been a long-standing problem [33]. The nature of the cosmological constant (CC) term introduced by Einstein to his field equation is *a priori* undefined. On the other hand, the quantum vacuum energy satisfies the properties of the CC, yet its value is about 124 orders of magnitude larger than the critical density of the Universe, which is comparable to what is required for the CC to explain the observed accelerating expansion of the Universe [34,35]. Because of mathematical considerations, the CC term cannot be removed from the Einstein-Hilbert action. The SKYC equation is second order in terms of the affine connection. While the metric is *a priori* not a dynamical variable, in order to reduce the SKYC equation to the Einstein equation one must define the relation between the connection and the metric in the usual way. As a result the SKYC equation can be redressed as a third-order differential equation of the metric where the CC term is necessarily absent. The CC term, however, is recovered as an integration constant in reducing the SKYC equation to the Einstein equation. In this approach the CC is no longer arbitrary, but rather is determined by the boundary condition of the Universe, which is geometrical in nature and has nothing to do with the quantum vacuum energy. For example, it has been proposed that the underlying geometry of the Universe is de Sitter and this integration constant is associated with its radius of curvature [20,31,36]. The SKYC formulation of gravity may therefore provide a solution to the CC problem.

Aside from the CC problem, recent observational data suggests the possible existence of "dark radiation" (DR), which is conventionally parametrized by $N_{\rm eff}$ [37–41]. The standard model predicts $N_{\rm eff} = 3.04$ at the epoch where the Universe was dominated by photons and neutrinos after the electron-positron annihilation. However, recent experimental data suggests a larger $N_{\rm eff}$ [37–41]. It has been suggested that the difference, $\Delta N_{\rm eff} \equiv N_{\rm eff} - 3 \simeq 1$ at nearly the 2σ level, may indicate the existence of an additional, heretofore unobserved DR density. On the theoretical side, this additional radiation was also called upon by the braneworld-inspired and other cosmological models [42–48]. It happens that another integration constant in the SKYC theory has exactly the required character of dark radiation. We note however—as has been pointed out recently by Birrell *et al.* [49]—that the fact that neutrinos have rest mass and that their distributions are nonthermal under free-steaming can well explain such an increase of $N_{\rm eff}$ without the need to invoke dark radiation. With this in mind, the dark radiation arising from SKYC gravity is nonetheless a free bonus at our disposal subject to observational constraints.

The purpose of this paper is to examine explicitly the salient features—in particular the CC and dark radiation—of the SKYC gravity mentioned above. To do so, we derive the Friedmann-Lemaître-Robertson-Walker (FLRW) equation associated with the SKYC gravity with integration constants. We identify one integration constant as the CC and the other as dark radiation. Subsequently, we develop a method to investigate the property of dark radiation and demonstrate that the density of dark radiation is a constant on the constant-time hypersurface in every metric.

This paper is arranged as follows. In Sec. II, we review the SKYC gravity. In the next section, the SKYC field equation on the FLRW metric is derived and investigated. We discuss the property of nullity in Camenzind's matter field in the perfect fluid model. In Sec. IV, we characterize the properties of the dark radiation. In Sec. V, we summarize and discuss our findings. In the appendices, we discuss the problem with Cook's theory, which intends to give rise to the SKYC equation from the action level.

II. SKYC EQUATION

In a non-Abelian gauge theory, the field strength is defined as

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu], \tag{1}$$

where $D_\mu = \partial_\mu - igA_\mu$ and A_μ is the gauge potential. The field strength $F_{\mu\nu}$ can therefore be expressed in terms of the gauge potential A_ν as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{2}$$

When connecting Riemmanian geometry with the non-Abelian gauge theory, the Christoffel symbol, $\Gamma^{\lambda}_{\mu\nu}$, and the Riemann tensor, $R_{\mu\nu\sigma\lambda}$, play the role of the gauge potential A_μ and the field strength $F_{\mu\nu}$, respectively. Motivated by this gauge connection, Stephenson, Kilmister, and Newman investigated an alternative theory of gravity and obtained the following equation of motion (EOM):

$$\nabla_\nu R^{\nu}{}_{\mu\alpha\beta} = 0, \tag{3}$$

which is a higher-order derivative with respect to the metric [12,13]. Invoking the second Bianchi identity, it can be readily verified that Eq. (3) is equivalent to

$$\nabla_\mu R_{\alpha\beta} - \nabla_\beta R_{\alpha\mu} = 0. \tag{4}$$

In fact, Eq. (3) is exactly Yang's gravitational field equation for pure space under the $GL(n)$ gauge symmetry [4].

To complete the gravitational field equation, Kilmister introduced the matter current I such that [13]

$$\nabla_\alpha R_{\beta\mu} - \nabla_\beta R_{\alpha\mu} = I_{\alpha\beta\mu}. \tag{5}$$

He found that I has 20 independent components under the constraint of the conservation law,

$$\nabla_\mu I^{\alpha\beta\mu} = 0. \tag{6}$$

However, he did not provide the explicit form for the current density I.

Later, Camenzind wrote down a Yang-Mills field equation for SO(3,1) with a current J that was different from Kilmister's I,

$$\nabla_\nu R^{\nu}{}_{\mu\alpha\beta} = 8\pi G J_{\alpha\beta\mu}, \tag{7}$$

where the current density $J^{\alpha\beta\mu}$ has the form

$$\begin{aligned} J^{\alpha\beta\mu} &= \nabla^\beta (T^{\mu\alpha} - 1/2 g^{\mu\alpha} T^{\lambda}{}_{\lambda}) \\ &\quad - \nabla^\alpha (T^{\mu\beta} - 1/2 g^{\mu\beta} T^{\lambda}{}_{\lambda}). \end{aligned} \tag{8}$$

Here $T^{\mu\alpha}$ is the energy-momentum tensor [19]. Nevertheless, an action associated with this matter current remained lacking. Inspired by the analogy with Maxwell's theory, Cook later proposed an action term for the matter current, $\Gamma^{\mu}_{\alpha\beta} J^{\alpha\beta}_{\mu}$, from which the SKYC field equation can be derived [20]. It can be verified that solutions of Eq. (7) cover the entire solution space of the Einstein equation with source [50].

III. THE SKYC EQUATION IN THE FLRW METRIC

In this section we will use the FLRW metric to derive a modified Friedmann equation from the SKYC field equation (7), and discuss the nullity of Camezind's current density in the radiation-dominated case.

A. EOM in a homogeneous universe

We consider a homogeneous and isotropic universe and use the FLRW metric to study Eq. (7). The metric is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{9}$$

where $a(t)$ is the scale factor. From the symmetry property of the FLRW metric, the energy-momentum tensor $T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{10}$$

where $u^\mu = (1, 0, 0, 0)$. In Eq. (7), we assume that $T_{\mu\nu}$ satisfies the conservation law,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{11}$$

which in turn leads to the usual equation

$$\dot\rho = -3H(\rho + p), \tag{12}$$

where $H \equiv \dot a / a$.

The only nontrivial components of Eq. (7) are

$$\nabla_\nu R^{\nu}{}_{i0i} = 8\pi J_{0ii}, \tag{13}$$

where no summation over the repeated index "i" occurs. Other components vanish due to the symmetry of the metric. The expression for $T_{\mu\nu}$, Eq. (10), leads to

$$a^2 \dddot a + a\dot a \ddot a - 2\dot a^3 - 2k\dot a = 8\pi G\left(\frac{1}{2} a(\dot\rho - \dot p) + \dot a(\rho + p)\right), \tag{14}$$

which can also be expressed as

$$\ddot H + 4H\dot H - 2k\frac{\dot a}{a^3} = 8\pi G\left(\frac{1}{2}(\dot\rho - \dot p) + H(\rho + p)\right). \tag{15}$$

Equation (15) is the modified Friedmann equation in SKYC theory. By integrating Eq. (15) with Eq. (12), we can arrive at

$$\dot H + 2H^2 + \frac{k}{a^2} = \frac{8\pi G}{2}\left(\frac{1}{3}\rho - p\right) + 2C_1, \tag{16}$$

where C_1 is an integration constant. By performing another integration on Eq. (16), one obtains

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + C_1 + \frac{C_2}{a^4}. \tag{17}$$

This is the Friedmann equation for the SKYC theory.

Equation (17) can be interpreted as follows. In the FLRW universe, C_1 plays the role of the CC term. As will be shown in the following section, the constant C_1 plays the same role as the CC even in an inhomogeneous universe; this fact has been shown by Cook [20]. (Our C_1 is identical to the trace part of $X_{\mu\nu}$ in his article.) The term involving C_2, on the other hand, scales the same as radiation. Since it does not couple with any matter except gravity, it can be identified as the dark radiation. We emphasize that the constant of integration C_2 originates from the lhs of Eq. (7), not Camenzind's current density. Specifically, the C_2 term is a vacuum solution in the SKYC theory. We note that although the C_2 term is not induced by the matter current, it does affect the evolution of the universe.

B. Nullity in the current density tensor

In this subsection we show that Camenzind's current density associated with homogeneous radiational fields must be null in the FLRW universe. This, however, is not inconsistent with the basic notions of the radiation-dominant Universe. Actually, the integration constant C_2 can play the role of the sum of the ordinary radiation in standard cosmology and the dark radiation.

For our purpose, we decompose the energy-momentum tensor $T_{\mu\nu}$ as

$$T_{\mu\nu} = T^{(\mathrm{R})}_{\mu\nu} + T^{(\mathrm{CC})}_{\mu\nu} + T^{(\mathrm{other})}_{\mu\nu}, \tag{18}$$

where $T^{(\mathrm{R})}_{\mu\nu}$, $T^{(\mathrm{CC})}_{\mu\nu}$ and $T^{(\mathrm{other})}_{\mu\nu}$ are the energy-momentum tensors for radiation, the cosmological constant, and the sum of all other contributions, respectively. Each energy-momentum tensor can be written in the same form as in Eq. (10) with $p = w\rho$, where w equals $1/3$ for radiation, 0 for the cosmological constant, and any value for the sum of the other contributions, respectively. Since Camenzind's current density $J_{\alpha\beta\gamma}$ comprises $T_{\mu\nu}$, it can be decomposed into

$$J_{\alpha\beta\gamma} = J^{(\mathrm{R})}_{\alpha\beta\gamma} + J^{(\mathrm{CC})}_{\alpha\beta\gamma} + J^{(\mathrm{other})}_{\alpha\beta\gamma}. \tag{19}$$

Substituting $T_{\mu\nu}$ into Eq. (8), one can easily verify that $T^{(R)}_{\mu\nu}$ and $T^{(CC)}_{\mu\nu}$ do not affect $J^{\alpha\beta\gamma}$. That is, $J^{(R)}_{\alpha\beta\gamma}$ and $J^{(CC)}_{\alpha\beta\gamma}$ are null. This means that radiation and the cosmological constant do not couple with gravity in the FLRW background.

The nullity is apparently inconsistent with Eq. (17) because ρ in Eq. (17) represents the sum of all energy densities, including radiation and the cosmological constant. A possible resolution to this inconsistency would be to re-express Eq. (17) as

$$H^2 = \frac{8\pi G}{3}\rho^{(\mathrm{other})} - \frac{k}{a^2} + \tilde{C}_1 + \frac{\tilde{C}_2}{a^4}, \tag{20}$$

where $\rho^{(\mathrm{other})}$ is the energy density derived from $T^{(\mathrm{other})}_{\mu\nu}$, while there are no contributions by $T^{(\mathrm{R})}_{\mu\nu}$ and $T^{(\mathrm{CC})}_{\mu\nu}$ because of the nullity. While the new constants of integration, $\tilde{C}_1$ and $\tilde{C}_2$, follow the same a dependences as those of $\rho^{(\mathrm{CC})}$ and $\rho^{(\mathrm{R})}$ in GR, respectively, their values need not be the same as C_1 and C_2. Therefore, extracting the a dependence of $\rho^{(\mathrm{R})}$ as $\rho^{(\mathrm{R})} = E/a^4$, we can decompose $\tilde{C}_1$ and $\tilde{C}_2$ as $\tilde{C}_1 = C_1 + \Lambda$ and $\tilde{C}_2 = C_2 + E$. As a result of the decomposition, Eq. (20) can be transformed back to

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho^{(\mathrm{other})} - \frac{k}{a^2} + C_1 + \Lambda + \frac{C_2 + E}{a^4} \\ &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + C_1 + \frac{C_2}{a^4}, \end{aligned} \tag{21}$$

i.e., the same form as that in the original Eq. (17). We therefore conclude that Eq. (17) is exact even though the original equation (7) is null for radiation and the cosmological constant.

We see that in spite of the nullity of radiation and the cosmological constant, the SKYC theory can still provide a consistent FLRW equation. Even so, the fine-tuning problem remains. From observations, we know that the integration constants C_1 and C_2 must be tiny and therefore their values need to be fine-tuned. Nevertheless, this fine-tuning problem of the cosmological constant is milder than that in the usual GR. In GR the cosmological constant is *a priori* arbitrary and is generally identified with quantum vacuum energy, which is inevitably much larger than the critical energy density of the Universe, resulting in the infamous 124 orders of magnitude discrepancy between the two. On the other hand, in the SKY approach C_1 is an integration constant subjected to the boundary condition of the Universe, which is apparently unrelated to the quantum vacuum energy. Thus the new fine-tuning problem may not be as stringent as that in GR. In addition to the fine-tuning of C_1, it is also necessary to fine-tune $\tilde{C}_2$, which may be even more unnatural than that of C_1. Observations show that the value of $\tilde{C}_2$ should be comparable to that of E. This seems to imply that the two are actually related.

IV. CHARACTERISTICS OF DARK RADIATION

In the previous section, we have seen the fact that one cannot separate the effect of the dark radiation from that of the ordinary radiation in the homogeneous universe. In order to distinguish them, we consider the case of an inhomogeneous universe.

First, we solve Eq. (7) in the inhomogeneous universe. We define the effective energy-momentum tensor as

$$\hat{T}_{\mu\nu} \equiv \frac{1}{8\pi G}G_{\mu\nu} - T_{\mu\nu}. \tag{22}$$

Because of Eq. (11), the divergence of this gives

$$\nabla^\mu \hat{T}_{\mu\nu} = 0. \tag{23}$$

Substituting Eq. (23) into Eq. (7), we have

$$\nabla_\alpha \hat{T}_{\beta\gamma} - \nabla_\beta \hat{T}_{\alpha\gamma} - \frac{1}{2}(g_{\beta\gamma}\nabla_\alpha \hat{T} - g_{\alpha\gamma}\nabla_\beta \hat{T}) = 0, \tag{24}$$

where $\hat{T}$ is the trace of $\hat{T}_{\mu\nu}$. Multiplying $g^{\alpha\gamma}$ by Eq. (24), we can obtain

$$\nabla_\beta \hat{T} = 0, \tag{25}$$

where we use Eq. (23). Integrating this, we have

$$\hat{T} = 4\Lambda_{\rm eff}, \tag{26}$$

where $\Lambda_{\rm eff}$ is a constant. As we have commented at the end of Sec. III A, it plays exactly the role of the CC.

We now separate the contributions other than the effective CC from $\hat{T}_{\mu\nu}$,

$$S_{\mu\nu} \equiv \hat{T}_{\mu\nu} - \Lambda_{\rm eff} g_{\mu\nu}. \tag{27}$$

We know from Eqs. (23) and (24) that the following equations must be satisfied:

$$S = 0, \tag{28}$$

$$\nabla^\mu S_{\mu\nu} = 0, \tag{29}$$

$$\nabla_\alpha S_{\beta\gamma} - \nabla_\beta S_{\alpha\gamma} = 0. \tag{30}$$

If the form of $S_{\mu\nu}$ is the same as that of the radiation fluid on the FLRW metric, then it can be shown that these equations are satisfied. Therefore, $S_{\mu\nu}$ is related to the C_2 term.

In order to see the difference of the effective energy-momentum tensor $S_{\mu\nu}$ from a real radiation fluid, we analyze its property on a general metric in the form of a fluid,

$$S_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{31}$$

with

$$u_\mu u^\mu = -1. \tag{32}$$

The traceless condition (28) fixes the relation between ρ and p as

$$\rho = \frac{1}{3} p. \tag{33}$$

Then $S_{\mu\nu}$ becomes

$$S_{\mu\nu} = \frac{4}{3}\rho u_\mu u_\nu + \frac{1}{3}\rho g_{\mu\nu} \tag{34}$$

$$= \rho u_\mu u_\nu + \frac{1}{3}\rho h_{\mu\nu}, \tag{35}$$

where

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{36}$$

is the induced metric on the hypersurface which is orthogonal to u^μ.

Equation (29) can be written as

$$\frac{4}{3} u_\beta u^\mu \partial_\mu \rho + \frac{1}{3}\partial_\beta \rho + \frac{4}{3}\rho (u_\mu \nabla^\mu u_\beta + u_\beta \nabla^\mu u_\mu) = 0. \tag{37}$$

Multiplying u^β and $h_\nu{}^\beta$ by the above expression, respectively, we can obtain

$$u^\mu \partial_\mu \rho = -\frac{4}{3}\rho \nabla^\mu u_\mu, \tag{38}$$

$$h_\nu{}^\beta \partial_\beta \rho = -4\rho u_\mu \nabla^\mu u_\nu, \tag{39}$$

where we use

$$u^\beta \nabla^\mu u_\beta = \frac{1}{2}\nabla^\mu (u^\beta u_\beta) = 0, \tag{40}$$

$$h_\nu{}^\beta \nabla^\mu u_\beta = (\delta_\nu{}^\beta + u_\nu u^\beta)\nabla^\mu u_\beta = \nabla^\mu u_\nu. \tag{41}$$

Multiplying u_ν by Eq. (38) and subtracting it from Eq. (39), we have

$$\partial_\nu \rho = \frac{4}{3}\rho u_\nu \nabla^\mu u_\mu - 4\rho u_\mu \nabla^\mu u_\nu. \tag{42}$$

On the other hand, Eq. (30) can be written as

$$u_\beta u_\gamma \partial_\alpha \rho - u_\alpha u_\gamma \partial_\beta \rho + \frac{1}{3} h_{\beta\gamma}\partial_\alpha \rho - \frac{1}{3} h_{\alpha\gamma}\partial_\beta \rho + \frac{4}{3}\rho (u_\beta \nabla_\alpha u_\gamma - u_\alpha \nabla_\beta u_\gamma - u_\gamma \nabla_\alpha u_\beta + u_\gamma \nabla_\beta u_\alpha) = 0. \tag{43}$$

Multiplying $u^\gamma u^\beta$ by Eq. (43), we have

$$h_\alpha{}^\beta \partial_\beta \rho - \frac{4}{3}\rho u^\beta \nabla_\beta u_\alpha = 0. \tag{44}$$

Combining it with Eq. (39), we have

$$u_\mu \nabla^\mu u_\nu = 0, \tag{45}$$

$$h_\alpha{}^\beta \partial_\beta \rho = 0. \tag{46}$$

Multiplying $h_\lambda{}^\gamma u^\beta$ by Eq. (43), we have

$$-\frac{1}{3} h_{\lambda\alpha} u^\beta \partial_\beta \rho - \frac{4}{3}\rho \nabla_\alpha u_\lambda = 0, \tag{47}$$

where we have used Eq. (45). Combining it with Eq. (38), we find

$$\nabla_\alpha u_\lambda = \frac{1}{3} h_{\alpha\lambda} \nabla^\mu u_\mu. \tag{48}$$

Multiplying $u^\gamma h_\lambda{}^\beta$ and $h_\epsilon{}^\gamma h_\lambda{}^\beta$ by Eq. (43), respectively, gives

$$u_\alpha h_\lambda{}^\beta \partial_\beta \rho + \frac{4}{3}\rho \nabla_\alpha u_\lambda - \frac{4}{3}\rho \nabla_\lambda u_\alpha - \frac{4}{3} u_\lambda u^\beta \nabla_\beta u_\alpha = 0, \tag{49}$$

$$\frac{1}{3} h_{\epsilon\lambda}\partial_\alpha \rho - \frac{1}{3} h_{\epsilon\alpha} h_\lambda{}^\beta \partial_\beta \rho - \frac{4}{3}\rho u_\alpha \nabla_\lambda u_\epsilon = 0. \tag{50}$$

Substituing Eqs. (42), (45), (46), and (48) into Eqs. (49) and (50), we see that they are automatically satisfied and no additional condition is obtained.

In summary, we have transcribed the original equations for $S_{\mu\nu}$ into Eqs. (38), (45), (46), and (48), and

$$\partial_\nu \rho = \frac{4}{3}\rho u_\nu \nabla^\mu u_\mu, \tag{51}$$

which govern the characteristics of ρ and u_μ. In turn, these conditions constrain the form of the effective energy-momentum tensor. Equation (39) means that the energy density must be constant on the hypersurface that is orthogonal to u^μ. Therefore, the dark radiation from the effective energy-momentum tensor can affect only the background dynamics and we confirm the existence of the real radiation fluid from its perturbation.

V. CONCLUSION

We investigated the SKYC theory of gravity by way of solving its field equation, Eq. (7), in an FLRW universe and arrived at a modified SKYC Friedmann equation, Eq. (15). Being a higher-order derivative equation than that for GR, the SKYC Firedmann equation gives rise to two integration constants when it is reduced to a lower order. One of the two is clearly related to the cosmological constant, while the other is related to the dark radiation. We have also demonstrated, in a homogeneous universe, that this dark radiation is indistinguishable from the ordinary radiation. In addition, we pointed out the nullity of the current density J in the radiation case under the FLRW metric.

In order to further pin down the nature of our dark radiation, we turned to a general, inhomogeneous universe and introduced a methodology to look for its possible differences from ordinary radiation. We solved Eq. (7) in the inhomogeneous universe and constrained the form of the tensor $S_{\mu\nu}$ in Sec. IV. We found that if ρ is fixed at one point, then it will be a constant on the hypersurface orthogonal to u^μ. This means that there does not exist any degree of freedom for the perturbed dark radiation. That is, this SKYC dark radiation only has the zero-mode term and no perturbed term. In contrast, the ordinary radiation can be perturbed and can therefore propagate in all of spacetime. We conclude that the SKYC dark radiation is indeed different from the ordinary one. We would like to comment, however, that in our derivation in Sec. IV, the expression $S_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$ is not the most general form. We will pursue a more general expression for it in our future work.

Some comments are in order with regard to the relationship between the SKYC dark radiation and the inflation. The dark radiation C_2 term is an integration constant in this theory. That is, it is determined by the initial or boundary conditions of the Universe. If the SKYC theory is incorporated with inflation, then the density of the dark radiation $\rho_{\rm DR}$ must start from a tiny value. Otherwise the inflation can not be trigged because $\rho_{\rm DR}$ scales as a^{-4} and dominates at early times. At the end of inflation and after $\sim$60 e-foldings, the scale factor has grown by $\sim 10^{20}$ times. This means that $\rho_{\rm DR}$ must be smaller by 80 orders of magnitude than $\rho_{\rm DR}$ at late times. Being so tiny, we may as well set C_2 to zero. On the other hand, if SKYC theory does not include the inflation scenario, then the C_2 term can in principle be identified with the dark radiation and be fixed by the observational data. The recent Planck data gives $N_{\rm eff} = 3.36^{+0.68}_{-0.64}(95\%)$ based on the combination of WMAP + highL data [41]. However, this fit produces a 2.5 s.d. tension with direct astrophysical measurements of the Hubble constant. Including priors from supernova surveys removes this tension and results in $N_{\rm eff} = 3.62^{+0.50}_{-0.48}(95\%)$. The larger $N_{\rm eff}$ suggests a need for the dark radiation. Without combining SKYC gravity with inflation, we fix C_2 with the Planck data and find $C_2 \sim 5.59 \times 10^{-32}\ {\rm kg/m^3}$, while when including supernova priors the Planck data gives $C_2 \sim 1.02 \times 10^{-31}\ {\rm kg/m^3}$.

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APPENDIX A: DIFFERENCE BETWEEN STEPHENSON'S AND YANG'S APPROACHES

Stephenson's gravitational equation is similar to Yang's, but there are some important differences between the two theories. In brief, Stephenson's theory covers less solutions than Yang's even if the connection in Stephenson's theory is identified as the Christoffel symbol. We will discuss this point in more detail in this appendix.

In the Palatini formalism, the affine connection and the metric are treated as independent variables and the derived EOMs can determine the relation between the two. For example, the relation between the connection and the metric can be identified as the condition for the metric compatibility when applying the Palatini formalism to derive EOMs from the Einstein-Hilbert action. Stephenson and Cook obtained EOMs by applying the Palatini formalism to the quadratic curvature Lagrange density for pure gravity without matter. Regarding the matter field in the SKYC theory, Cook introduced his current density tensor at the action level. However, there are problems with his approach, which we will comment on in Appendix B.

In Yang's theory, the gravitational force is described by the $GL(4)$ gauge field b^a_μ. Here the latin letters stand for the indices of the $GL(4)$ gauge group. The action is

$$S_b = \int dx^4 \sqrt{-g}(g^{\mu\alpha} g^{\nu\beta} C^d_{ac} C^c_{bd} f^a_{\mu\nu} f^b_{\alpha\beta}), \tag{A1}$$

where $f^a_{\mu\nu}$ is the field strength,

$$f^a_{\mu\nu} = b^a_{\mu,\nu} - b^a_{\nu,\mu} - C^a_{bc} b^b_\mu b^c_\nu, \tag{A2}$$

and C^a_{bc} is the structure constant.

In order to connect his gauge field b^a_μ to the metric, Yang introduced a higher-order curvature term in the action. The final form of his gravitational action is

$$S_g(b^a_\mu, g^{\mu\nu}) = \int d^4x\sqrt{-g}(g^{\mu\alpha}g^{\nu\beta}C^d_{ac}C^c_{bd}f^a_{\mu\nu}f^b_{\alpha\beta} - R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}). \tag{A3}$$

Performing variations of the action with respect to the gauge field b^a_μ and the metric, one arrives at two types of equations of motion. The index a in b^a_μ has 4×4 values and it can be redefined as $a \equiv \{kl\}$, where k and l run from 0 to 3. Yang used an ansatz $b^{\{kl\}}_\alpha \delta_{k\mu}\delta_{l\nu} = \{^{\mu}_{\nu\alpha}\}$ that satisfies the equation derived from the variation of the action with respect to the metric. It gives the relation

$$f^{(\alpha\beta)}_{\mu\nu} = -R^\alpha{}_{\beta\mu\nu}, \tag{A4}$$

and the other equations become

$$\nabla_\beta R_{\mu\alpha} = \nabla_\alpha R_{\mu\beta}. \tag{A5}$$

This is Yang's gravitational equation based on the gauge theory and it covers the solutions of Einstein's equation for pure space, that is, without matter [14].

In Stephenson's theory, there are two EOMs which stem from the variations of the action with respect to the metric and the connection, respectively [12],

$$-R^{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} + R^{\alpha\mu\beta\gamma}R_{\alpha\nu\beta\gamma} + 2R^{\alpha\beta\mu\gamma}R_{\alpha\beta\nu\gamma} - \frac{1}{2}g^\mu_\nu R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 0, \tag{A6}$$

$$\nabla_\alpha(R^{\nu\sigma\alpha}_\mu\sqrt{-g}) = 0. \tag{A7}$$

The relation between the metric and the connection, however, is different from that in GR. In particular, the affine connection in his theory can in principle be different from the Levi-Civita connection. If one identifies the affine connection as the Levi-Civita connection in Stephenson's theory, the resulting EOMs are not equivalent to Yang's. Then, Stephenson's equations become

$$R^{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{4}g^\mu_\nu R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 0, \tag{A8}$$

$$\nabla_\alpha R^{\nu\sigma\alpha}_\mu = 0. \tag{A9}$$

These equations can be recognized as Yang's gravitational equation [Eq. (A8)] under the additional constraint of Eq. (A9). In this sense, the solution space of the resultant Stephenson's equations must be a subset of Yang's gravitational equation. Without specifying the connection as the Levi-Civita connection, Stephenson's equations should in principle have different solutions than Yang's.

APPENDIX B: PROBLEMS WITH COOK'S THEORY

Cook introduced the matter action to Stephenson's theory. His recipe, however, not only retains the original problem of Stephenson's theory but also introduces another one. The action he proposed is

$$S_G = \frac{-1}{16\pi}\int_\Sigma (R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} + 16\pi J_\mu{}^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta})\sqrt{-g}d^4x, \tag{B1}$$

where the tensor $J_\mu{}^{\alpha\beta}$ is Cook's current density, which has the same form as Camenzind's. This matter action term, however, is not general covariant because the connection is not a covariant tensor [51].

The lack of general covariance must be closely related to the nonconservation of the current density J. According to Noether's theorem, a symmetry property of the action always goes hand-in-hand with the conservation of the current. Therefore, it seems impossible to construct a general-covariant action based on Camenzind's current density. This suggests that one should search for a different form of the current density other than that of Camenzind's, under the constraint that GR must be recovered. We will investigate this further.

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