Negative off-diagonal conductivities in a weakly coupled quark-gluon plasma at the leading-log order

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We calculate the conductivity matrix of a weakly coupled quark-gluon plasma at the leading-log order. By setting all quark chemical potentials to be identical, the diagonal conductivities become degenerate and positive, while the off-diagonal ones become degenerate but negative (or zero when the chemical potential vanishes). This means a potential gradient of a certain fermion flavor can drive backward currents of other flavors. A simple explanation is provided for this seemingly counterintuitive phenomenon. It is speculated that this phenomenon is generic even beyond the leading-log order, and it is most easily measured in cold atom experiments.

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I. INTRODUCTION

Hydrodynamics describes the evolution of a fluid perturbed away from thermal equilibrium by long wavelength fluctuations. The long wavelength physics (long compared with the mean free path of particle collisions) can be systematically described by an expansion of space-time derivatives on classical fields with prefactors called transport coefficients. These transport coefficients encode the physics of short (compared with the mean free path) distance and are inputs to hydrodynamics. But they can be computed, in principle, once the microscopic theory of the system is known.

We are interested in computing the transport coefficients in QCD with N_f flavors of massless quarks at finite temperature (T) and chemical potentials (μ_a, $a = 1, 2, \ldots, N_f$). The leading transport coefficients at the first derivative order include the shear viscosity (η), bulk viscosity (ζ), and the conductivity matrix (λ).

The shear viscosity of QCD has attracted a lot of attention recently. Its ratio with the entropy density (s) extracted from the hot and dense matter created at the Relativistic Heavy Ion Collider [1–4] just above the phase transition temperature (T_c) yields $1/(4\pi) \leq \eta/s \leq 2.5/(4\pi)$ at $T_c \leq T \leq 2T_c$ [5], which is close to a conjectured universal lower bound of $1/(4\pi)$ [6] inspired by the gauge/gravity duality [7–9]. This value of η/s cannot be explained by extrapolating the perturbative QCD result [10–13]. The smallest η/s is likely to exist near T_c [14,15] (see, e.g., Ref. [13] for a compilation and more references). Also, finite μ results suggests that η/s is smaller at smaller μ. This is based on results of perturbative QCD at $T \gg T_c$ [16] and of a hadronic gas at $T \ll T_c$ and small μ [17]. It is speculated that the same pattern will persist at T_c such that the smallest η/s might exist near T_c with $\mu = 0$ [16].

For the bulk viscosity, the sum rule study [18,19] shows that ζ increases rapidly near T_c when T approaches T_c from above. This is consistent with the lattice gluon plasma result near T_c [20] and perturbative QCD result [21] at much higher T. This, when combined with pion gas results below T_c [22–26], suggests that ζ/s has a local maximum near T_c (see, e.g., Ref. [13] for a compilation). Unlike η/s, the perturbative QCD result shows very small μ dependence in ζ/s [16]. Note that, at high μ, there are also bulk viscosities governed by the weak interaction such as the Urca processes, which have consequences in neutron star physics [27–32]. These are quite different from the transport coefficients from the strong interaction mentioned above.

The perturbative QCD calculations of η and ζ with finite μ were performed at the leading-log (LL) order of the strong coupling constant (g) expansion in Ref. [16]. Corrections at the leading order is only suppressed by a factor of $\log^{-1} g^{-1}$, thus our results only hold when T or μ in the calculation is much larger than Λ_{QCD}, the scale on which QCD becomes nonperturbative. Also, the calculation is not applicable to the color superconducting phase at $\mu/T \to \infty$, since the vacuum in the calculation has no symmetry breaking.

In this work, we apply the same perturbative QCD approach to compute the conductivity matrix λ at the LL order. The conductivity is an important transport coefficient that plays an essential role in the evolution of electromagnetic fields in heavy ion collisions [33–37].

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The conductivity in strongly coupled quark gluon plasma was calculated with lattice QCD in the quenched approximation [38] or with dynamical quark flavors [39,40] and the Dyson–Schwinger equation [41].

We first review the constraints from the second law of thermal dynamics (i.e., the entropy production should be non-negative) which shows that the particle diffusion, heat conductivities, and electric conductivity are all unified into one single conductivity in this system. When $N_f > 1$, the conductivity becomes an $N_f \times N_f$ matrix. We then show through the Boltzmann equation that the conductivity matrix λ at the LL order is symmetric and positive definite ($\sum_{a,b}\lambda_{ab}X^aX^b > 0$ for any real, non-vanishing vector X). The former is a manifestation of the Onsager relation, while the latter is a manifestation of the second law of thermal dynamics.

For simplicity, we show the numerical results of λ with all fermion chemical potential to be identical. In this limit, there are only two independent entries in λ. All the diagonal matrix elements are degenerate and positive since λ is positive definite. However, the off-diagonal matrix elements are degenerate but negative at finite μ. This means a gradient $\boldsymbol{\nabla}\mu_a$ can drive a current of flavor a along the gradient direction, but it will also drive currents of different flavors in the opposite direction. A simple explanation is provided for this seemingly counterintuitive backward current phenomenon. It is speculated that this phenomenon is generic even beyond the leading-log order, and it is most easily measured in cold atom experiments.

II. ENTROPY PRINCIPLE IN HYDRODYNAMICS

A. Single-flavor case

Let us start from the hydrodynamical system with only one flavor of quark of electric charge Q. The energy-momentum conservation and current conservation yield

$$\partial_\mu T^{\mu\nu} = QF^{\nu\lambda}j_\lambda, \qquad \partial_\mu j^\mu = 0, \tag{1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor, j^μ is the quark current, and $F^{\nu\lambda}$ is the electromagnetic field strength tensor. The long wavelength physics can be systematically described by the expansion of space-time derivatives

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{(0)} + \varepsilon T^{\mu\nu}_{(1)} + \varepsilon^2 T^{\mu\nu}_{(2)} + \cdots, \\ j^\mu &= j^\mu_{(0)} + \varepsilon j^\mu_{(1)} + \varepsilon^2 j^\mu_{(2)} + \cdots, \end{aligned} \tag{2}$$

where we have used the parameter ε to keep track of the expansion and we will set $\varepsilon = 1$ at the end. $F^{\nu\lambda}$ is counted as $\mathcal{O}(\varepsilon)$. We will then assume the system is isotropic and homogeneous in thermal equilibrium so there are no special directions or intrinsic length scales macroscopically. We also assume the underlying microscopic theory satisfies parity, charge conjugation, and time reversal symmetries such that the antisymmetric tensor $\varepsilon^{\mu\nu\alpha\beta}$ does not contribute to $T^{\mu\nu}$ and j^μ. Also, we assume the system is fluidlike, describable by one (and only one) velocity field (the conserved charged is assumed to be not broken spontaneously; otherwise, the superfluid velocity needs to be introduced as well). Also, at $\mathcal{O}(\varepsilon^0)$, the system is in local thermal equilibrium; i.e., the system is in equilibrium in the comoving frame in which the fluid velocity is zero. With these assumptions, we can parametrize

$$\begin{aligned} T^{\mu\nu}_{(0)} &= (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}, \qquad j^\mu_{(0)} = nu^\mu, \\ T^{\mu\nu}_{(1)} &= \Pi(u^\mu u^\nu - g^{\mu\nu}) + \pi^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu, \\ j^\mu_{(1)} &= \nu^\mu, \end{aligned} \tag{3}$$

where $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$ and ϵ, P, and n are the energy density, pressure, and number density, respectively. The fluid velocity $u^\mu = (u^0, \boldsymbol{u}) = (1, \boldsymbol{v})/\sqrt{1-\boldsymbol{v}^2}$ and $u^\mu u_\mu = 1$. Π, $\pi^{\mu\nu}$, h^μ and ν^μ are the bulk viscous pressure, shear viscous tensor, heat flow vector, and diffusion current. They satisfy the orthogonal relations, $\pi^{\mu\nu}u_\nu = \nu^\mu u_\mu = h^\mu u_\mu = 0$.

The covariant entropy flow is given by [42,43]

$$\begin{aligned} S^\mu &= \beta P u^\mu + \beta T^{\mu\nu}u_\nu - \beta\mu j^\mu \\ &= su^\mu + \beta h^\mu - \beta\mu\nu^\mu, \end{aligned} \tag{4}$$

where $\beta = 1/T$ and $s = \beta(\epsilon + P - \mu n)$ is the entropy density. Taking the space-time derivative of S^μ, then using the Gibbs–Duhem relation $d\epsilon = Tds + \mu dn$ and the conservation equations (1), we obtain the equation for entropy production,

$$\begin{aligned} \partial_\mu S^\mu = &-\nu^\mu[\partial_\mu(\beta\mu) + \beta QE_\mu] + h^\mu(\partial_\mu\beta + \beta u^\nu\partial_\nu u_\mu) \\ &+ \beta\pi^{\mu\nu}\partial_{\langle\mu}u_{\nu\rangle} - \beta\Pi\,\partial\cdot u, \end{aligned} \tag{5}$$

where the symmetric traceless tensor $\partial_{\langle\mu}u_{\nu\rangle}$ is defined by

$$\partial_{\langle\mu}u_{\nu\rangle} = \frac{1}{2}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right]\partial^\alpha u^\beta \tag{6}$$

and where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and $E^\mu \equiv F^{\mu\nu}u_\nu$ is the electric field in the comoving frame.

At $\mathcal{O}(\varepsilon)$, $\partial_\mu T^{\mu\nu} \simeq \partial_\mu T^{\mu\nu}_{(0)} = 0$. This equation yields

$$\partial_\mu\beta + \beta u^\nu\partial_\nu u_\mu = \frac{n}{\epsilon + P}[\partial_\mu(\beta\mu) + \beta QE_\mu], \tag{7}$$

where we have used the thermodynamic equation $dP = \beta(\epsilon + P)dT + nTd(\beta\mu)$. This identity simplifies Eq. (5) to

$$\begin{aligned} \partial_\mu S^\mu = &-\left(\nu^\mu - \frac{n}{\epsilon + P}h^\mu\right)[\partial_\mu(\beta\mu) + \beta QE_\mu] \\ &+ \beta\pi^{\mu\nu}\partial_{\langle\mu}u_{\nu\rangle} - \beta\Pi\,\partial\cdot u. \end{aligned} \tag{8}$$

The second law of thermodynamics requires $\partial_\mu S^\mu \geq 0$. It can be satisfied if, up to terms orthogonal to $\partial\cdot u$, $\partial_{\langle\mu}u_{\nu\rangle}$

and $[\partial_\mu(\beta\mu) + \beta Q E_\mu]$, Π, $\pi^{\mu\nu}$, h^μ, and ν^μ have the forms at $\mathcal{O}(\varepsilon)$,

$$\pi^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle},$$
$$\Pi = -\zeta\partial\cdot u, \tag{9}$$
$$\nu^\mu - \frac{n}{\epsilon + P}h^\mu = \lambda\Delta^{\mu\nu}[\partial_\nu(\beta\mu) + \beta Q E_\nu]$$

where $\Delta^{\mu\nu}$ is inserted because $\nu^\mu u_\mu = h^\mu u_\mu = 0$. The coefficients η, ζ, and λ are transport coefficients with names of shear viscosity, bulk viscosity, and conductivity, respectively. The second law of thermodynamics requires these transport coefficients to be non-negative.

On the right-hand side of Eq. (9), the three vectors $\partial_\nu\mu$, $\partial_\nu\beta$, and E_ν form a unique combination and share the same transport coefficient λ [42]. It is obtained by assuming $\partial_\mu T^{\mu\nu}_{(0)} = 0$ and $T^{\mu\nu}_{(0)}$ has the ideal fluid form described in Eq. (3). In general, we do not expect this to be true in all systems (e.g., a solid might not have the ideal fluid description), and hence there could be more transport coefficients. Conventionally, the transport coefficients corresponding to $\partial_\nu\mu$, $\partial_\nu\beta$, and E_ν are called particle diffusion, heat conductivity, and electric conductivity, respectively.

In hydrodynamics, the choice of the velocity field is not unique. One could choose $\mathbf{u}$ to align with the momentum density $T^{0i}\hat{\mathbf{i}}$ or the current $\mathbf{j}$, or their combinations. However, the system should be invariant under the transformation $u_\mu \to u'_\mu = u_\mu + \varepsilon\delta u_\mu$ as long as $u'^2_\mu = 1$ is maintained [or $u^\mu\delta u_\mu = 0$ at $\mathcal{O}(\varepsilon)$]. Under this transformation, $h_\mu \to h'_\mu = h_\mu + (\epsilon + P)\delta u_\mu$ and $\nu_\mu \to \nu'_\mu = \nu_\mu + n\delta u_\mu$. However, the entropy production equation (8) remains invariant under this transformation.

In this paper, we will be working in the Landau frame with $\mathbf{u}$ proportional to the momentum density $T^{0i}\hat{\mathbf{i}}$ such that $T^{0i} = 0$ in the comoving frame. Then,

$$\mathbf{h} = 0, \qquad \boldsymbol{\nu} = \lambda[-\boldsymbol{\nabla}(\beta\mu) + \beta Q\mathbf{E}] \tag{10}$$

from Eq. (9). λ is positive; the sign makes sense for particle diffusion and electric conduction because the diffusion is from high to low density, and positively charged particles move along the $\mathbf{E}$ direction. However, heat conduction induces a flow from low to high temperature. This result is counterintuitive. This is because $\boldsymbol{\nabla}T$ induces a momentum flow $\mathbf{h}$. If we choose to boost the system to the Landau frame in which $\mathbf{h} = 0$, then the physics is less transparent. For particle diffusion and electric conduction, this is not a problem because one could have particles and antiparticles moving in opposite directions and still keep the net momentum flow zero.

The physics of heat conduction becomes clear in the Eckart frame in which $\mathbf{u}$ is proportional to the current $\mathbf{j}$ and we have

$$\boldsymbol{\nu} = 0, \qquad \mathbf{h} = -\frac{\epsilon + P}{n}\lambda[-\boldsymbol{\nabla}(\beta\mu) + \beta Q\mathbf{E}]. \tag{11}$$

In this frame, the direction of heat conduction is correct (while the physics of particle diffusion and electric conduction become less transparent). As expected, $\mathbf{h}$ stays finite when $\mu = Q = 0$ but $\boldsymbol{\nabla}\beta \neq 0$.

B. Multiflavor case

When the flavor of massless quarks is increased to N_f, then there are N_f conserved currents j^μ_a (the conserved electric current is just a combination of them). The hydrodynamical equations become

$$\partial_\mu T^{\mu\nu} = \sum_{a=1}^{N_f} Q_a F^{\nu\lambda} j_{a,\lambda},$$
$$\partial_\mu j^\mu_a = 0, \qquad a = 1, 2, \ldots, N_f. \tag{12}$$

Then, the entropy production yields

$$\partial_\mu S^\mu = -\sum_{a=1}^{N_f}\left(\nu^\mu_a - \frac{n_a}{\epsilon + P}h^\mu\right)[\partial_\mu(\beta\mu_a) + \beta Q_a E_\mu]$$
$$+ \beta\pi^{\mu\nu}\partial_{\langle\mu}u_{\nu\rangle} - \beta\Pi\partial\cdot u \geq 0. \tag{13}$$

Working in the Landau frame, we have

$$\boldsymbol{\nu}_a = \sum_{b=1}^{N_f}\lambda_{ab}[-\boldsymbol{\nabla}(\beta\mu_b) + \beta Q_b\mathbf{E}]. \tag{14}$$

Our task is to compute the λ matrix, which can be achieved by setting $\boldsymbol{\nabla}(\beta\mu_b) \neq 0$ but $\mathbf{E} = 0$. The second law of thermodynamics dictates λ being a positive definite matrix.

III. EFFECTIVE KINETIC THEORY

We will use the Boltzmann equation to compute our LL result of λ. It has been shown that the Boltzmann equation gives the same leading-order result as the Kubo formula in the zcoupling constant expansion in a weakly coupled ϕ^4 theory [44,45] and in hot QED [46], provided the leading T and μ dependence in particle masses and scattering amplitudes are included. This conclusion is expected to hold in perturbative QCD as well [47].

The Boltzmann equation of a quark-gluon plasma describes the evolution of the color- and spin-averaged distribution function $\tilde{f}^i_p(x)$ for particle i ($i = g$, q_a, $\bar{q}_a$ with $a = 1.2\ldots N_f$ for gluon, N_f quarks, and N_f antiquarks),

$$\frac{d\tilde{f}^i_p(x)}{dt} = \tilde{\mathcal{C}}_i, \tag{15}$$

where $\tilde{f}^i_p(x)$ is a function of space-time $x^\mu = (t, \mathbf{x})$ and momentum $p^\mu = (E_p, \mathbf{p})$.

For the LL calculation, we only need to consider two-particle scattering processes denoted as $c_1c_2 \to c_3c_4$.

The collision term has the form

$$C_{c_1c_2\to c_3c_4} \equiv \int_{k_1k_2k_3} d\Gamma_{c_1c_2\to c_3c_4}[\tilde{f}^{c_1}_{k_1}\tilde{f}^{c_2}_{k_2}\tilde{F}^{c_3}_{p}\tilde{F}^{c_4}_{k_3} - \tilde{F}^{c_1}_{k_1}\tilde{F}^{c_2}_{k_2}\tilde{f}^{c_3}_{p}\tilde{f}^{c_4}_{k_3}], \tag{16}$$

where $\tilde{F}^g = 1 + \tilde{f}^g$ and $\tilde{F}^{q(\bar{q})} = 1 - \tilde{f}^{q(\bar{q})}$ and

$$d\Gamma_{c_1c_2\to c_3c_4} = \frac{1}{2E_p}|M_{c_1c_2\to c_3c_4}|^2 \prod_{i=1}^{3}\frac{d^3k_i}{(2\pi)^3 2E_{k_i}} \times (2\pi)^4\delta^{(4)}(k_1+k_2-k_3-p), \tag{17}$$

where $|M_{c_1c_2\to c_3c_4}|^2$ is the matrix element squared with all colors and helicities of the initial and final states summed over. The scattering amplitudes can be regularized by hard thermal loop propagators, and, in this paper, we use the same scattering amplitudes as in Ref. [11] (see also Table I of Ref. [16]). Then, the collision term for a quark of flavor a is

$$N_q\tilde{\mathcal{C}}_{q_a} = \frac{1}{2}C_{q_aq_a\leftrightarrow q_aq_a} + C_{q_a\bar{q}_a\leftrightarrow q_a\bar{q}_a} + \frac{1}{2}C_{gg\leftrightarrow q_a\bar{q}_a} + C_{q_ag\leftrightarrow q_ag} + \sum_{b,b\neq a}^{N_f-1}(C_{q_aq_b\leftrightarrow q_aq_b} + C_{q_a\bar{q}_b\leftrightarrow q_a\bar{q}_b} + C_{q_b\bar{q}_b\leftrightarrow q_a\bar{q}_a}), \tag{18}$$

where $N_q = 2\times 3 = 6$ is the quark helicity and color degeneracy factor and the factor $1/2$ is included when the initial state is formed by two identical particles. Similarly,

$$N_g\tilde{\mathcal{C}}_g = \frac{1}{2}C_{gg\leftrightarrow gg} + \sum_{a=1}^{N_f}(C_{gq_a\leftrightarrow gq_a} + C_{g\bar{q}_a\leftrightarrow g\bar{q}_a} + C_{q_a\bar{q}_a\leftrightarrow gg}), \tag{19}$$

where $N_g = 2\times 8 = 16$ is the gluon helicity and color degeneracy factor. In equilibrium, the distributions are denoted as $f^{q_a(\bar{q}_a)}$ and f^g, with

$$f^g_p = \frac{1}{e^{u\cdot p/T}-1}, \tag{20}$$

$$f^{q_a(\bar{q}_a)}_p = \frac{1}{e^{(u\cdot p\mp\mu_a)/T}+1}, \tag{21}$$

where T is the temperature, u is the fluid four velocity, and μ_a is the chemical potential for the quark of flavor a. They are all space-time dependent.

The thermal masses of gluon and quark/antiquark for external states (the asymptotic masses) can be computed via [47,48]

$$m_g^2 = \sum_i N_iC_i\frac{2g^2}{d_A}\int\frac{d^3p}{(2\pi)^3 2E_p}f^i_p, \tag{22}$$

$$m_q^2 = m_{\bar{q}}^2 = 2C_Fg^2\int\frac{d^3p}{(2\pi)^3 2E_p}(2f^g_p + f^q_p + f^{\bar{q}}_p), \tag{23}$$

where $d_A = 8$, $C_g = C_A = 3$, and $C_{q(\bar{q})} = C_F = 4/3$. This yields

$$m_g^2 = \frac{C_A}{6}g^2T^2 + \sum_{a=1}^{N_f}\frac{C_F}{16}g^2\left(T^2 + \frac{3}{\pi^2}\mu_a^2\right), \tag{24}$$

$$m_{q_a}^2 = \frac{1}{4}C_Fg^2\left(T^2 + \frac{\mu_a^2}{\pi^2}\right), \tag{25}$$

where we have set $E_p = |\mathbf{p}|$ in the integrals on the right-hand sides of Eqs. (22) and (23). The difference from nonvanishing masses is of higher order. In this work, we only need the fact that the thermal masses are proportional to g^2 for the LL results.

A. Linearized Boltzmann equation

Matching to the derivative expansion in hydrodynamics, we expand the distribution function of particle i as a local equilibrium distribution plus a correction,

$$\tilde{f}^i_p(x) = f^i_p - \varepsilon f^i_p(1\mp f^i_p)\chi^i, \tag{26}$$

where the upper/lower sign corresponds to the femion/boson distribution. Inserting Eq. (26) into Eq. (15), we can solve the linearized Boltzmann equation by keeping linear terms in space-time derivatives. Here, we neglect the viscous terms related to $\partial_\mu u_\nu$ in χ^i and consider only the $\nabla(\beta\mu_a)$ terms.

At the zeroth order, $\mathcal{O}(\varepsilon^0)$, the system is in local thermal equilibrium, and the Boltzmann equation (15) is satisfied, $\tilde{\mathcal{C}}[f^i_p] = 0$. At $\mathcal{O}(\varepsilon)$, the left-hand side of the Boltzmann equation yields

$$\frac{df^g_p}{dt} = -\beta f^g_pF^g_p\sum_{a=1}^{N_f}\left[\frac{n_aT}{\epsilon+P}\mathbf{p}\cdot\nabla(\beta\mu_a)\right], \tag{27}$$

and

$$\frac{df^{q_a(\bar{q}_a)}_p}{dt} = -\beta f^{q_a(\bar{q}_a)}_pF^{q_a(\bar{q}_a)}_p \times\sum_{b=1}^{N_f}\left(\frac{n_bT}{\epsilon+P}\mp\frac{T}{E^{q_a(\bar{q}_a)}_p}\delta_{ab}\right)\mathbf{p}\cdot\nabla(\beta\mu_b). \tag{28}$$

To derive this result, we have used $\partial_\mu u^0 = 0$ in the local fluid rest frame, in which $u^\mu = (1,0,0,0)$ and $\partial_\mu T^{\mu\nu}_{(0)} = 0$ and $\partial_\mu j^\mu_{a(0)} = 0$, which yields

$$\frac{\partial\epsilon}{\partial t} = -(\epsilon+p)\nabla\cdot\mathbf{u} \qquad \frac{\partial\mathbf{u}}{\partial t} = -\frac{\nabla P}{\epsilon+p} \tag{29}$$

and

$$\frac{\partial n_a}{\partial t} = -n_a \nabla \cdot \mathbf{u}. \tag{30}$$

And then, by applying thermodynamic relations, we can replace the time derivatives of T, μ, and $\mathbf{u}$ with spatial derivatives:

$$\begin{aligned} \frac{\partial T}{\partial t} &= -T\left(\frac{\partial P}{\partial \epsilon}\right)_n \nabla \cdot \mathbf{u}, \\ \frac{\partial \mu}{\partial t} &= -\left[\mu\left(\frac{\partial P}{\partial \epsilon}\right)_n + \left(\frac{\partial P}{\partial n}\right)_\epsilon\right] \nabla \cdot \mathbf{u}, \\ \frac{\partial \mathbf{u}}{\partial t} &= -\beta \nabla T - \sum_{a=1}^{N_f} \frac{n_a T}{\epsilon + p} \nabla\left(\frac{\mu_a}{T}\right). \end{aligned} \tag{31}$$

Those relations lead to Eqs. (27) and (28).

To get the right-hand side of the Boltzmann equation at $\mathcal{O}(\varepsilon)$, we parametrize χ^i of Eq. (26) as

$$\chi^i = \beta \sum_{a=1}^{N_f} \mathbf{A}^{ia}(p) \cdot \nabla(\beta \mu_a), \qquad \mathbf{A}^{ia}(p) = A^{ia}(|\mathbf{p}|)\hat{\mathbf{p}}. \tag{32}$$

The matrix A^{ia} is $(2N_f + 1) \times N_f$. We will see there are $(2N_f + 1) \times N_f$ equations to constrain them.

For each Boltzmann equation, we have a linear combination of N_f terms of $\nabla(\beta \mu_a)$. Since each $\nabla(\beta \mu_a)$ is linearly independent to each other, there are N_f equations for each Boltzmann equation. Totally, we have $2N_f + 1$ Boltzmann equations; thus, we have $(2N_f + 1) \times N_f$ equations to solve for A^{ia}. These equations are

$$\frac{n_a \mathbf{p}}{\epsilon + P} = \beta \frac{1}{f_p^g F_p^g} \frac{1}{N_g}\left[\frac{1}{2}\mathbf{C}^a_{gg\to gg} + \sum_{c=1}^{N_f}(\mathbf{C}^a_{gq_c\to gq_c} + \mathbf{C}^a_{g\bar{q}_c\to g\bar{q}_c} + \mathbf{C}^a_{q_c\bar{q}_c\to gg})\right], \tag{33}$$

$$\begin{aligned} \left(\frac{n_a}{\epsilon + P} - \frac{1}{E_p^{q_b}}\delta_{ab}\right)\mathbf{p} &= \frac{\beta}{f_p^{q_b} F_p^{q_b}} \frac{1}{N_q}\left[\frac{1}{2}\mathbf{C}^a_{q_b q_b\to q_b q_b} + \mathbf{C}^a_{q_b\bar{q}_b\to q_b\bar{q}_b} + \frac{1}{2}\mathbf{C}^a_{gg\to q_b\bar{q}_b} + \mathbf{C}^a_{q_b g\to q_b g} \right. \\ &\quad \left. + \sum_{c,c\neq b}^{N_f-1}(\mathbf{C}^a_{q_b q_c\to q_b q_c} + \mathbf{C}^a_{q_b\bar{q}_c\to q_b\bar{q}_c} + \mathbf{C}^a_{q_c\bar{q}_c\to q_b\bar{q}_b})\right], \end{aligned} \tag{34}$$

and

$$\begin{aligned} \left(\frac{n_a}{\epsilon + P} + \frac{1}{E_p^{\bar{q}_b}}\delta_{ab}\right)\mathbf{p} &= \frac{\beta}{f_p^{\bar{q}_b} F_p^{\bar{q}_b}} \frac{1}{N_q}\left[\frac{1}{2}\mathbf{C}^a_{\bar{q}_b\bar{q}_b\to\bar{q}_b\bar{q}_b} + \mathbf{C}^a_{\bar{q}_b q_b\to\bar{q}_b q_b} + \frac{1}{2}\mathbf{C}^a_{gg\to\bar{q}_b q_b} + \mathbf{C}^a_{\bar{q}_b g\to\bar{q}_b g} \right. \\ &\quad \left. + \sum_{c,c\neq b}^{N_f-1}(\mathbf{C}^a_{\bar{q}_b\bar{q}_c\to\bar{q}_b\bar{q}_c} + \mathbf{C}^a_{\bar{q}_b q_c\to\bar{q}_b q_c} + \mathbf{C}^a_{\bar{q}_c q_c\to\bar{q}_b q_b})\right], \end{aligned} \tag{35}$$

where

$$\mathbf{C}^a_{c_1c_2\to c_3c_4}(p) \equiv \int_{k_1k_2k_3} d\Gamma_{c_1c_2\to c_3c_4} f^{c_1} f^{c_2} F^{c_3} F^{c_4}[\mathbf{A}^{c_1a}(k_1) + \mathbf{A}^{c_2a}(k_2) - \mathbf{A}^{c_3a}(k_3) - \mathbf{A}^{c_4a}(p)]. \tag{36}$$

Formally, we can rewrite these linearized Boltzmann equations in a compact form,

$$|\mathbf{S}^a\rangle = \mathcal{C}_\lambda |\mathbf{A}^a\rangle, \tag{37}$$

where $|\mathbf{S}^a\rangle$ and $|\mathbf{A}^a\rangle$ are both vectors of $(2N_f + 1)$ components and $\mathcal{C}_\lambda$ is a $(2N_f + 1) \times (2N_f + 1)$ matrix.

B. Conductivity matrix

In the kinetic theory, the quark current of flavor a is

$$j_a^\mu = N_q \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E^{q_a}}(\tilde{f}_p^{q_a} - \tilde{f}_p^{\bar{q}_a}). \tag{38}$$

Expanding this expression to $\mathcal{O}(\varepsilon)$ and matching it to Eq. (14), we have

$$\lambda_{ab} = \frac{N_q \beta}{3} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E^{q_a}}(f^{q_a} F^{q_a} \mathbf{p} \cdot \mathbf{A}^{q_a b} - f^{\bar{q}_a} F^{\bar{q}_a} \mathbf{p} \cdot \mathbf{A}^{\bar{q}_a b}). \tag{39}$$

Since we are working in the Landau frame, we should impose the Landau–Lifshitz condition

$$0 = T^{0j} = -\sum_i N_i \int \frac{d^3p}{(2\pi)^3} f^i F^i \chi^i p^j. \tag{40}$$

This implies

$$\sum_i N_i \int \frac{d^3p}{(2\pi)^3} f^i F^i \mathbf{p} \cdot \mathbf{A}^{ia} = 0. \tag{41}$$

We can use these constraints to rewrite Eq. (39) as

$$\lambda_{ab} = -\frac{\beta}{3}\Big\{N_g \int \frac{d^3p}{(2\pi)^3} f^g F^g \frac{n_a}{\epsilon + P}\mathbf{p}\cdot\mathbf{A}^{gb} + \sum_{c=1}^{N_f} N_q \int \frac{d^3p}{(2\pi)^3} f^{q_c} F^{q_c}\Big(\frac{n_a}{\epsilon+P} - \frac{1}{E^{q_a}}\delta_{ca}\Big)\mathbf{p}\cdot\mathbf{A}^{q_c b} + \sum_{c=1}^{N_f} N_q \int \frac{d^3p}{(2\pi)^3} f^{\bar{q}_c} F^{\bar{q}_c}\Big(\frac{n_a}{\epsilon+P} + \frac{1}{E^{q_a}}\delta_{ca}\Big)\mathbf{p}\cdot\mathbf{A}^{\bar{q}_c b}\Big\}. \tag{42}$$

This form can be schematically written as

$$\lambda_{ab} = \langle \mathbf{A}^b|\mathbf{S}^a\rangle = \langle \mathbf{A}^b|\mathcal{C}_\lambda|\mathbf{A}^a\rangle, \tag{43}$$

where we have used Eq. (37) for the second equality. More explicitly,

$$\lambda_{ab} = \frac{\beta^2}{24}\Big(D^{ab}_{gg\to gg} + \sum_{c=1}^{N_f} D^{ab}_{q_c q_c\to q_c q_c} + \sum_{c=1}^{N_f} D^{ab}_{\bar{q}_c\bar{q}_c\to\bar{q}_c\bar{q}_c}\Big) + \frac{\beta^2}{6}\sum_{c=1}^{N_f}(D^{ab}_{q_c\bar{q}_c\to gg} + D^{ab}_{gq_c\to gq_c} + D^{ab}_{g\bar{q}_c\to g\bar{q}_c} + D^{ab}_{q_c\bar{q}_c\to q_c\bar{q}_c}) + \frac{\beta^2}{12}\sum_{\substack{c,d=1\\c\neq d}}^{N_f}(D^{ab}_{q_c q_d\to q_c q_d} + D^{ab}_{\bar{q}_c\bar{q}_d\to\bar{q}_c\bar{q}_d} + 2D^{ab}_{q_c\bar{q}_d\to q_c\bar{q}_d} + 2D^{ab}_{q_c\bar{q}_c\to q_d\bar{q}_d}), \tag{44}$$

where

$$D^{ab}_{c_1c_2\to c_3c_4} \equiv \int\prod_{i=1}^{4}\frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \times |M_{c_1c_2\to c_3c_4}|^2 f^{c_1}_{k_1} f^{c_2}_{k_2} F^{c_3}_{k_3} F^{c_4}_{k_4}[\mathbf{A}^{c_1a}(k_1) + \mathbf{A}^{c_2a}(k_2) - \mathbf{A}^{c_3a}(k_3) - \mathbf{A}^{c_4a}(k_4)]\cdot[\mathbf{A}^{c_1b}(k_1) + \mathbf{A}^{c_2b}(k_2) - \mathbf{A}^{c_3b}(k_3) - \mathbf{A}^{c_4b}(k_4)]. \tag{45}$$

From Eq. (37), it is clear that, if

$$\mathbf{A}^{ia}_0(p) = \mathbf{p}, \tag{46}$$

then, from momentum conservation, this implies

$$\mathcal{C}_\lambda|\mathbf{A}^a_0\rangle = 0. \tag{47}$$

Those modes are called zero modes [denoted by the subscript 0 in Eq. (46)]. They would have been a problem for Eq. (43) unless $\langle \mathbf{S}^a|\mathbf{A}^a_0\rangle = 0$, but this is guaranteed from the total momentum conservation at $\mathcal{O}(\varepsilon)$,

$$\frac{d}{dt}\sum_i\int dp^3\mathbf{p}f^i_p(x) = 0, \tag{48}$$

and Eqs. (27) and (28). Thus, we can just solve for $|\mathbf{A}^a\rangle$ in Eq. (43) by discarding the zero modes.

From Eqs. (44) and (45), we can see easily that $\lambda_{ab} = \lambda_{ba}$. This is a manifestation of the Onsager relation, which appears when particle scattering is symmetric under the time-reversal transformation. We can also see that λ is positive definite.

IV. LEADING-LOG RESULTS WITH IDENTICAL CHEMICAL POTENTIALS

Now, we are ready to solve the conductivity matrix λ. Our strategy to solve for λ_{ab} is to make use of Eq. (43) to solve for $|\mathbf{A}^a\rangle$ from λ_{aa} (no summation over a). Once all the $|\mathbf{A}^a\rangle$ are obtained, λ_{ab} can be computed. Also, in solving for λ_{aa}, one can use the standard algorithm to systematically approach the answer from below [13]. The dependence on the strong coupling constant is similar to that in shear viscosity—it is inversely proportional to the scattering rate, which scales as $g^4\ln g^{-1}$ with the $\ln g^{-1}$ dependence coming from regularizing the collinear infrared singularity by the thermal masses of quarks or gluons. λ is of mass dimension 2; thus, we will present our result in the normalized conductivity

$$\tilde{\lambda} \equiv \lambda\beta^2 g^4 \ln g^{-1} \tag{49}$$

such that $\tilde{\lambda}$ is dimensionless and coupling constant independent.

For simplicity, we will concentrate on the linear response of a thermal equilibrium system with all fermion chemical potentials to be identical, i.e., $\mu_a = \mu$ for all a's, but each $\nabla(\beta\mu_a)$ could be varied independently. This symmetry makes all the diagonal matrix elements (denoted as λ_{qq}) identical and all the off-diagonal ones (denoted as $\lambda_{qq'}$) identical. λ_{qq} and $\lambda_{qq'}$ are even in μ (and so are $\tilde{\lambda}_{qq}$ and $\tilde{\lambda}_{qq'}$) because our microscopic interaction (in vacuum) is invariant under charge conjugation; thus, λ should be invariant under $\mu_a \to -\mu_a$.

It is easy to diagonalize λ. One eigenvalue is

$$\lambda_+/N_f \equiv \lambda_{qq} + (N_f - 1)\lambda_{qq'}, \tag{50}$$

corresponding to the conductivity of the flavor singlet total quark current (λ_+ is the total quark current conductivity)

$$\boldsymbol{\nu} = \sum_{a=1}^{N_f}\boldsymbol{\nu}_a = -\lambda_+\sum_{a=1}^{N_f}\frac{\nabla(\beta\mu_a)}{N_f}. \tag{51}$$

The other $(N_f - 1)$ eigenvalues are degenerate with the value

$$\lambda_- \equiv \lambda_{qq} - \lambda_{qq'}. \tag{52}$$

They are the conductivities of the flavor nonsinglet currents

$$\Delta\boldsymbol{\nu}_a \equiv \boldsymbol{\nu}_a - \boldsymbol{\nu}_1 = -\lambda_-\nabla[\beta(\mu_a - \mu_1)], \tag{53}$$

with $a = 2, \ldots, N_f$.

$\tilde{\lambda}_{qq}$ and $\tilde{\lambda}_{qq'}$ are shown as functions of $(\mu/T)^2$ in Fig. 1 for various N_f with $N_f \le 16$ such that the system is asymptotically free, while $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ are shown in

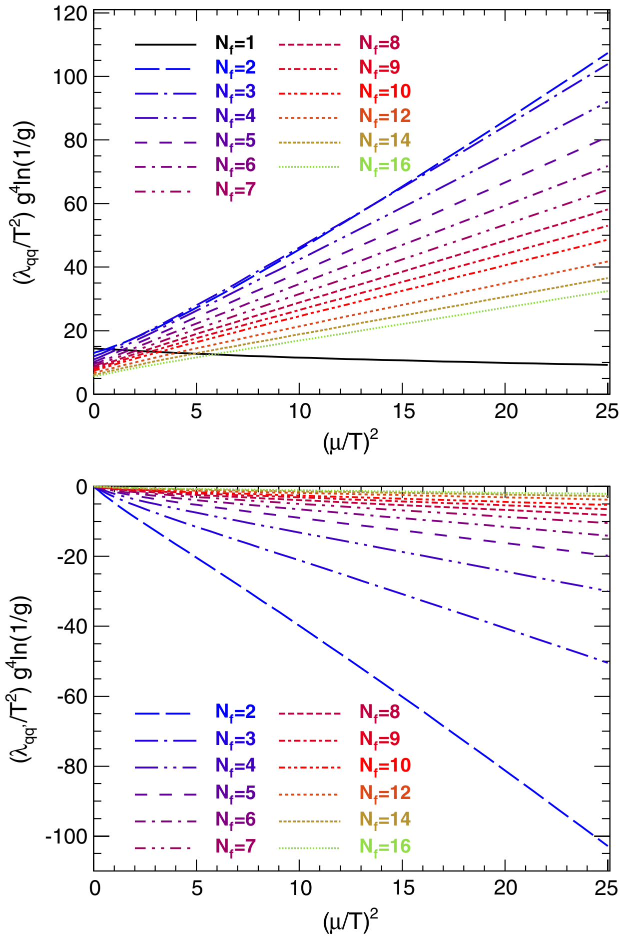

FIG. 1 (color online). The normalized diagonal conductivity $\tilde{\lambda}_{qq}$ (upper panel) and off-diagonal conductivity $\tilde{\lambda}_{qq'}$ (lower panel) as functions of $(\mu/T)^2$ for different N_f.

Fig. 2 (note that there is no $\tilde{\lambda}_{qq'}$ or $\tilde{\lambda}_-$ for $N_f = 1$). The fact that the matrix λ is positive definite makes $\tilde{\lambda}_{qq}$, $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ positive, but it imposes no constraint on the sign of $\tilde{\lambda}_{qq'}$.

When $(\mu/T)^2 \to 0$, we can expand $\tilde{\lambda}_{qq} = a_0 + a_1(\mu/T)^2 + \cdots$, and $\tilde{\lambda}_{qq'} = a'_0 + a'_1(\mu/T)^2 + \cdots$. We find $a'_0 = 0$ for all N_f, while the values of a_0, a_1, and a'_1 for different N_f are tabulated in Table. I. Our result for a_0 agrees within 0.1% to that of Arnold, Moore, and Yaffe calculated up to $N_f = 6$ listed in Table III of Ref. [10].

The $a'_0 = 0$ property is due to a bigger symmetry enjoyed by the LL results: if we just change all the quarks of flavor a into antiquarks while the rest of the system stays the same, then, as far as collision is concerned, the other quarks and the gluons will not feel any difference. This is because the LL result only depends on two-particle scattering, and although this action could change the sign of certain amplitudes, it does not change the collision rate. For example, the amplitudes of $q_a q_b \to q_a q_b$ and $\bar{q}_a q_b \to \bar{q}_a q_b$ $(a \neq b)$ have different signs because one of the couplings changes sign when we change the color into its anticolor, but the

TABLE I. The coefficients in the $(\mu/T)^2$ expansions of $\tilde{\lambda}_{qq}$ and $\tilde{\lambda}_{qq'}$ for small μ/T. Our result for $\mu/T = 0$ agrees within 0.1% to that of Arnold, Moore, and Yaffe, calculated up to $N_f = 6$ [10].

N_f	a_0	a_1	a'_1
1	14.3676	−0.3077	
2	12.9989	1.7347	−5.0372
3	11.8688	2.3969	−3.3569
4	10.9197	2.5757	−2.3922
5	10.1113	2.5680	−1.7906
6	9.4145	2.4791	−1.3909
7	8.8076	2.3600	−1.1117
8	8.2743	2.2319	−0.9090
9	7.8019	2.1076	−0.7572
10	7.3806	1.9880	−0.6404
11	7.0025	1.8766	−0.5487
12	6.6612	1.7731	−0.4754
13	6.3517	1.6791	−0.4159
14	6.0697	1.5917	−0.3668
15	5.8117	1.5121	−0.3260
16	5.5747	1.4384	−0.2916

amplitude squared is of the same. This makes the diagonal terms even in all the chemical potentials,

$$\lambda_{aa}(\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_{N_f}) = \lambda_{aa}(\mu_1, \mu_2, \ldots, \mu_{N_f}), \quad (54)$$

while the off-diagonal term λ_{ab} is odd in μ_a and μ_b but even in other chemical potentials,

$$\lambda_{ab}(\pm\mu_1, \ldots, \pm\mu_a, \ldots, \pm\mu_b, \ldots, \pm\mu_{N_f}) = \mathrm{sign}(\mu_a\mu_b)\lambda_{ab}(\mu_1, \ldots, \mp\mu_a, \ldots, \mp\mu_b, \ldots, \mu_{N_f}). \quad (55)$$

Thus, at the LL order, λ becomes diagonal when all the chemical potentials vanish.

To understand the other features of $\tilde{\lambda}_{qq}$ and $\tilde{\lambda}_{qq'}$, we first turn to $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ in the $(T/\mu)^2 \to 0$ limit. In this large chemical potential limit, the quark contribution dominates over those of the antiquark and gluon. The Fermi–Dirac distribution function f^{q_a} of quark q_a multiplied by its Pauli blocking factor F^{q_a} can be well approximated by a δ function, $f^{q_a}F^{q_a} \simeq T\delta(E_p - \mu)$.

We then first set $\nabla(\beta\mu_a) = \nabla(\beta\mu)$ for all a so all the currents $\boldsymbol{\nu}_a$ become identical. λ_+ can be rewritten as $\sum_{a,b}\lambda_{ab}$, and Eq. (42) yields

$$\lambda_+ \simeq -\frac{\beta}{3}N_q \int \frac{d^3p}{(2\pi)^3} T\delta(E_p - \mu)\mathbf{p}\cdot\mathbf{A} \times \sum_{a,b,c=1}^{N_f}\left(\frac{n_a}{\epsilon + P} - \frac{1}{\mu}\delta_{ca}\right). \quad (56)$$

The summation gives $N_f^2(\frac{n\mu}{\epsilon+P} - 1) \propto N_f^2 Ts/(\epsilon + P) \propto N_f^2T^2/\mu^2$ and $\lambda_+ \propto N_f^2T^2A$. On the other hand, Eq. (44)

gives $\lambda_+ \propto N_f^4 \mu^2 A^2$, where N_f^4 comes from summing the a, b, c, d indices of $D^{ab}_{q_c q_d \to q_c q_d}$ and we have used $f^{c_1}_{k_1} f^{c_2}_{k_2} F^{c_3}_{k_3} F^{c_4}_{k_4} \propto T^2$ in Eq. (45). These two conditions yield $\lambda_+ \propto N_f^0 T^4/\mu^2$. This is indeed what happens in Fig. 2 at large μ (although the $1/\mu^2$ dependence is not so obvious in this plot, but we have checked this at much larger μ^2/T^2).

We can perform the similar counting to the scaling of λ_-. From Eq. (42), $\lambda_- \propto N_f \mu^2 A$, and from Eq. (44), $\lambda_- \propto N_f^3 \mu^2 A^2$. Thus, $\lambda_- \propto \mu^2/N_f$, which is also observed in Fig. 2. The main difference in λ_+/N_f and λ_- is the T/μ dependence—λ_- has no cancellation factor of $(\frac{n\mu}{\epsilon+P} - 1) \propto T^2$ in large μ.

The different μ scaling between λ_+ and λ_- at large μ is due to fermion collisions (the contribution from antifermions and gluons can be neglected at large μ as mentioned above), which change the direction of the current and reduce the conductivity. While both flavor singlet and nonsinglet fermions can collide among themselves, they do not collide with each other (the scattering amplitude vanishes). This can be seen from the fermion scattering amplitude, which is proportional to $D_{\mu\nu}\langle f|j_+^\mu j_-^\nu|i\rangle$, with $D_{\mu\nu}$ the gluon propagator. We have set all the fermion chemical potentials to be identical; thus, both the initial and final states are flavor singlet, but the operator $j_+^\mu j_-^\nu$ is flavor nonsinglet. Therefore, the amplitude vanishes. On the other hand, this symmetry does not force the amplitudes from $j_+^\mu j_+^\nu$ or $j_-^\mu j_-^\nu$ matrix elements to vanish since both of the operators have flavor singlet components. More explicitly, we can work out the $N_f = 2$ case with u and d quarks as an example. $j_\pm^\mu = \bar{u}\gamma^\mu u \pm \bar{b}\gamma^\mu d$, $|i\rangle = |f\rangle = |ud - du\rangle/\sqrt{2}$, then $\langle f|j_+^\mu j_-^\nu|i\rangle = 0$, $\langle f|j_+^\mu j_+^\nu|i\rangle = -\langle f|j_-^\mu j_-^\nu|i\rangle \neq 0$. Thus, when μ, the flavor singlet chemical potential, is increased, the flavor singlet current experiences more collisions. Therefore, the flavor singlet conductivity λ_+ is reduced. For the flavor nonsinglet current, the increase of μ does not affect the collision. However, it will increase the averaged Fermi momentum such that the induced current and the flavor nonsinglet conductivity λ_- will be increased.

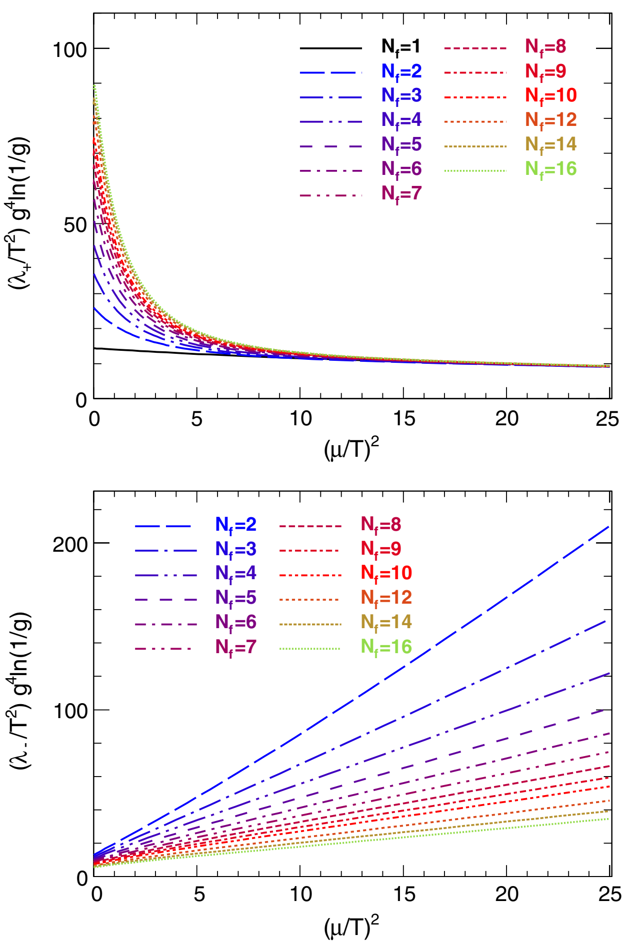

FIG. 2 (color online). The normalized conductivities $\tilde{\lambda}_+$ (upper panel) and $\tilde{\lambda}_-$ (lower panel) as functions of $(\mu/T)^2$ for different N_f.

Given the large μ behavior of $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$, the large μ behavior of $\tilde{\lambda}_{qq}$ and $\tilde{\lambda}_{qq'}$ is now easily reconstructed: $\tilde{\lambda}_{qq} \simeq (N_f - 1)\tilde{\lambda}_-/N_f \propto (N_f - 1)\mu^2/T^2 N_f^2$ $(N_f \geq 2)$ and $\tilde{\lambda}_{qq'} \simeq -\tilde{\lambda}_-/N_f \propto -\mu^2/T^2 N_f^2$. The sign of $\tilde{\lambda}_{qq'}$ can be best understood from the flavor nonsinglet current effect such that a gradient of μ_a induces anti-b currents $(b \neq a)$ and yields $\tilde{\lambda}_{qq'} < 0$. We can then interpolate $\tilde{\lambda}_{qq'}$ to $\tilde{\lambda}_{qq'} = 0$ at zero μ. There is no nontrivial structure at intermediate μ. For $\tilde{\lambda}_{qq}$, the $N_f = 1$ curve seems to be at odds with other N_f curves, but this anomaly disappears when viewed in the $\tilde{\lambda}_+$ plot.

The fact that $\lambda_{qq} > 0$ while $\lambda_{qq'} < 0$ at finite μ is intriguing. It means a gradient $\boldsymbol{\nabla}\mu_a$ can drive a current $\boldsymbol{\nu}_a$ along the $-\boldsymbol{\nabla}\mu_a$ direction, but it will also drive currents of different flavors in the opposite direction. This backward current phenomenon seems counterintuitive at first sight. But the physics behind it is just that the flavor singlet current experiences more collisions in a flavor singlet medium than the flavor nonsinglet ones. If the medium is flavor nonsinglet, e.g., $\mu_1 = -\mu_2 \neq 0$ while the other chemical potentials all vanish, then the flavor nonsinglet current $\boldsymbol{\nu}_2 - \boldsymbol{\nu}_1$ will experience more collisions than the flavor singlet current. (Using the same $N_f = 2$ example from above, the states are now $|i'\rangle = |f'\rangle = C_d|i\rangle = C_d|f\rangle$, with C_d a charge conjugation operator acting only on the d quarks. Then, $\langle f'|j_+^\mu j_-^\nu|i'\rangle = \langle f|C_d^\dagger j_+^\mu j_-^\nu C_d|i\rangle = \langle f|j_-^\mu j_+^\nu|i\rangle = 0$, and $\langle f'|j_+^\mu j_+^\nu|i'\rangle = \langle f|j_-^\mu j_-^\nu|i\rangle = -\langle f'|j_-^\mu j_-^\nu|i'\rangle = -\langle f|j_+^\mu j_+^\nu|i\rangle \neq 0$. Again, the flavor singlet and nonsinglet fermions can collide among themselves but not to each other.) Therefore, we will have $\lambda_{12} > 0$. This is consistent with Eq. (55) derived from the symmetry of the LL order along. Thus, the simple explanation based on collisions that we presented above seems quite generic. It might happen in other systems such as cold atoms as well. In that case, cold atom experiments might be the most promising ones to observe this backward current phenomenon.

V. SUMMARY

We have calculated the conductivity matrix of a weakly coupled quark-gluon plasma at the leading-log order. By setting all quark chemical potentials to be identical, the diagonal conductivities become degenerate and positive, while the off-diagonal ones become degenerate but negative (or zero when the chemical potential vanishes). This means a potential gradient of a certain fermion flavor can drive backward currents of other flavors. A simple explanation is provided for this seemingly counterintuitive phenomenon. It is speculated that this phenomenon is generic even beyond the leading-log order, and it is most easily measured in cold atom experiments.

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