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CAN $f(R)$ GRAVITY MIMIC GENERAL RELATIVITY?

JE-AN GU

Leung Center for Cosmology and Particle Astrophysics, National Taiwan University
Taipei 10617, Taiwan
jagu@ntu.edu.tw

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We discuss the stability of the general-relativity (GR) limit in modified theories of gravity, particularly the $f(R)$ theory. The problem of approximating the higher-order differential equations in modified gravity with the Einstein equations (2nd-order differential equations) in GR is elaborated. We demonstrate this problem with a heuristic example involving a simple ordinary differential equation. With this example we further present the iteration method that may serve as a better approximation for solving the equation, meanwhile providing a criterion for assessing the validity of the approximation. We then discuss our previous numerical analyses of the early-time evolution of the cosmological perturbations in $f(R)$ gravity, following the similar ideas demonstrated by the heuristic example. The results of the analyses indicated the possible instability of the GR limit that might make the GR approximation inaccurate in describing the evolution of the cosmological perturbations in the long run.

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1. Introduction

The $f(R)$ theory of modified gravity has been considered as a possible origin of the accelerating expansion of the universe, including the inflation at very early times (as an alternative to the inflaton field[1]) and the cosmic acceleration at the present epoch (as an alternative to dark energy: for viable models, see Refs. 2–5; for a review, see Ref. 6). On the one hand the correction to general relativity (GR) may be significant when driving the cosmic acceleration; on the other hand there should exist the GR limit in $f(R)$ gravity, as in any viable theory of modified gravity.

In particular, viable modified gravity should behave very similar to GR in the solar system for passing the local test of gravity, as well as at the cosmological scales at the early times relevant to the Big-Bang Nucleosynthesis (BBN) and the creation of the cosmic microwave background (CMB) photons in order to fit the BBN and CMB data. Accordingly, the viable modified gravity models are usually so constructed that the gravity action therein can be very close to the GR action in the above-mentioned circumstances.

Nevertheless, the existence of the GR limit in the action does not guarantee that in the solution, while what one truly needs is the GR limit in the solution because it is the solution that directly gives the theoretical prediction of observables. That is, even if there exists the GR limit at the action level, it may still be problematic to use the Einstein equations to approximate the true gravitational field equation at that GR limit of modified gravity. In particular, in GR the gravitational field equations, i.e. the Einstein equations, are 2nd-order partial differential equations (PDEs), while in many modified gravity theories (e.g., $f(R)$ gravity) they are higher-order PDEs. For such modified gravity theories, taking the GR limit at the field equation level and at the solution level is to approximate the higher-order PDEs with the 2nd-order PDEs. The validity of such an approximation requires verification.

2. A Heuristic Example

To demonstrate this issue, here we consider a toy example with a simple 2nd-order ordinary differential equation:

$$\epsilon\ddot{x}(t) + b\dot{x}(t) + cx(t) = 0, \tag{1}$$

where ϵ, b and c are constants, $\epsilon > 0$, $bc \neq 0$, and for simplicity $|b|, |c| \sim \mathcal{O}(1)$. The issue is as follows. When $\epsilon \ll |b|, |c|$, can this 2nd-order differential equation be well approximated by the following simpler 1st-order differential equation?

$$b\dot{x}(t) + cx(t) = 0. \tag{2}$$

The answer to this question can be clearly presented with the exact solutions:

$$x(t) = C_+ e^{\alpha_+ t} + C_- e^{\alpha_- t},\ \alpha_\pm = \frac{-b \pm \sqrt{b^2 - 4c\epsilon}}{2\epsilon} \quad \text{for Eq. (1)}, \tag{3}$$

$$x(t) = C_0 e^{-ct/b} \quad \text{for Eq. (2)}, \tag{4}$$

where $C_\pm$ and C_0 are integration constants, $\alpha_+ \cong -c/b - (c^2/b^3)\epsilon$ and $\alpha_- \cong -b/\epsilon + c/b$. When using Eq. (2) to approximate Eq. (1), one takes into consideration only the solution in Eq. (4). As compared to the solution in Eq. (3), this approximation ignores:

(i) the α_- mode ($C_- e^{\alpha_- t}$),

(ii) the factor $\exp[-(c^2/b^3)\epsilon t]$ in the α_+ mode ($C_+ e^{\alpha_+ t}$).

The validity of ignoring the α_- mode depends on the values of b and c. When $b > 0$, the α_- mode decreases much faster than the α_+ mode and therefore can be ignored. On the other hand, when $b < 0$, the α_- mode increases much faster than the α_+ mode and therefore cannot be ignored unless the coefficient C_- is set to zero by fine-tuning the initial condition. Regarding the factor $\exp[-(c^2/b^3)\epsilon t]$ in the α_+ mode, it cannot be ignored when $|(c^2/b^3)\epsilon t| \gtrsim \mathcal{O}(1)$, i.e., when one considers long-time evolution. This toy example demonstrates that the validity of approximating a higher-order differential equation with a lower-order differential equation requires careful examination.

An approximation better than Eqs. (2) and (4) is via iteration that, as to be shown, approximates the α_+ mode better but still ignores the α_- mode. Divide $x(t)$ into two parts of different orders of magnitude, $x(t) = x_0(t) + \epsilon x_1(t)$, and then recast Eq. (1) order by order as

$$b\dot{x}_0(t) + cx_0(t) = 0, \tag{5}$$

$$b\dot{x}_1(t) + cx_1(t) = -\epsilon\ddot{x}_0, \tag{6}$$

where we have neglected the term $\epsilon^2\ddot{x}_1$. The x_0 solution is the same as that in Eq. (4). Substituting the x_0 solution into the differential equation of x_1, one obtains the x_1 solution and then the approximate solution for $x(t)$:

$$x(t) = C_0 e^{-ct/b}\left(1 - \frac{c^2}{b^3}\epsilon t\right). \tag{7}$$

When $|(c^2/b^3)\epsilon t| \ll 1$, this is a good approximation of the α_+ mode. Thus, the approximate solution obtained via iteration may be good in describing the moderate-time evolution if the other mode can be ignored. Moreover, as an important advantage, the iteration method can provide a criterion ($|(c^2/b^3)\epsilon t| \ll 1$) for assessing the validity of the approximation, in contrast to the approximation in Eqs. (2) and (4) that gives no such criterion.

3. Stability of the General-Relativity Limit in $f(R)$ Gravity

In $f(R)$ gravity the correction to GR is represented by a function of the Ricci scalar:

$$S_g = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}[R + f(R)]. \tag{8}$$

When the function $f(R)$ is constant, the action goes back to the GR action with a cosmological constant. In the viable $f(R)$ models of the late-time cosmic acceleration, $f(R)$ is close to a constant for large R (corresponding to the local scales and to the cosmological scales at early times), while it may significantly vary for small R (corresponding to the cosmological scales at late times). Accordingly, it is conventional to treat GR as a good approximation for prescribing the early-time evolution of the cosmic expansion and the cosmic structures. Nevertheless, as pointed out above, whether GR is a good approximation requires careful examination. In Ref. 7 we examined the accuracy of the GR approximation by investigating the early-time evolution of the cosmological perturbations in $f(R)$ gravity. The results indicated a slight instability of the GR limit that may make the GR approximation inaccurate in the long run.

To demonstrate the GR approximation to $f(R)$ gravity and its potential problem for the early-time evolution of cosmological perturbations, here we simply present one of the evolution equations: a differential equation of the metric perturbation in the following simplified format.

$$\dot{f}_R \dot{\eta} = \sum \text{GR-terms} + \sum f\text{-terms}, \tag{9}$$

where $f_R \equiv df/dR$, the overhead dot denotes the derivative with respect to the conformal time τ in the Robertson-Walker metric, and η is one of the two scalar metric perturbations in the Fourier space, $h(\vec{k},\tau)$ and $\eta(\vec{k},\tau)$, defined by the line element:

$$ds^2 = a^2(\tau)\left\{-d\tau^2 + \left[\delta_{ij} + h_{ij}(\vec{x},\tau)\right] dx^i dx^j\right\}, \tag{10}$$

and the Fourier integral:

$$h_{ij}(\vec{x},\tau) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \left[\hat{k}_i \hat{k}_j h(\vec{k},\tau) + \left(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij}\right) 6\eta(\vec{k},\tau)\right]. \tag{11}$$

The "f-terms" denote the terms proportional to the derivatives of f, and the "GR-terms" are the other terms that also appear in the corresponding equation in GR. At early times, $\dot{f}_R\dot{\eta}$ and the f-terms are much smaller than the GR-terms. One may accordingly invoke the GR approximation where $\dot{f}_R\dot{\eta}$ and the f-terms are ignored, and then treat Eq. (9) as a constraint equation of the GR-terms: $\sum$ GR-terms $= 0$.

To examine the GR approximation, in Ref. 7 we invoked a better approximation for solving the evolution equations of cosmological perturbations including Eq. (9). Note that Eq. (9) can hardly be numerically solved because of the numerical instability caused by the smallness of the factor $\dot{f}_R$ on the left-hand side. The basic principle of our method is similar to the iteration demonstrated by the heuristic example in the previous section. In our approximation we divided η into two parts of different orders of magnitude, η_0 and η_1, and then derived their evolution equations that can be solved numerically, meanwhile providing a criterion for assessing the validity of the approximation. After solving the evolution equations, as a result we found that the GR approximation may be good with the accuracy better than 1% before the redshift $z \sim 10^{1.5} \sim 30$, but may not be accurate enough in the longer run.

This issue is analogous to that in the standard cosmology caused by the tight coupling between photons and baryons (before the recombination around $z \sim 10^3$) that renders the evolution equations of the perturbations difficult to solve, particularly those involving the term $an_e\sigma_T(\theta_b - \theta_\gamma)$, where n_e is the electron number density, σ_T the cross section of the Thomson scattering, and θ the divergence of the fluid velocity defined by $(\rho + P)\theta(\vec{k},\tau) \equiv ik^j \delta T^0{}_j(\vec{k},\tau)$. Before recombination the largeness of the Thomson opacity, $an_e\sigma_T \equiv \tau_c^{-1}$, causes instability in the numerical solution of the evolution equations. A rough approximation is to set $\theta_b = \theta_\gamma$, i.e., the comoving of the baryon and photon fluids, which is analogous to setting $\sum$ GR-terms $= 0$ in Eq. (9) by ignoring $\dot{f}_R\dot{\eta}$ and the f-terms. It was pointed out that the comoving approximation with $\theta_b = \theta_\gamma$ is not accurate enough. Accordingly the "tight-coupling approximation," a more accurate approximation with the basic principle similar to iteration, was proposed and invoked to account for the slip between the photon and baryon fluids.[8]

4. Summary

In this paper we have discussed the instability of the GR limit in modified gravity, particularly in the $f(R)$ theory. In viable modified gravity theories the existence of the GR limit is requisite. In addition to the GR limit at the action level, the existence and the stability of the GR limit at the solution level need to be verified, particularly when the gravitational field equations in modified gravity are higher-order differential equations.

We have heuristically demonstrated the problem (of approximating a higher-order with a lower differential equation) via a simple ordinary differential equation. This toy example shows that in general such approximation is problematic. With this example we have also presented the iteration method that may give a better approximate solution and provide a criterion for assessing the validity of the approximation.

In Ref. 7 we investigated the early-time evolution of the cosmological perturbations in $f(R)$ gravity and carefully examined the validity and the accuracy of the GR approximation. We invoked an iteration method as a better approximation that facilitates the numerical analysis of the early-time evolution of the cosmological perturbations, meanwhile providing a criterion for assessing both the validity of the GR approximation and that of our approximation with iteration. As a result, we found hints indicating the instability of the GR limit that may make the GR approximation inaccurate in describing long-time evolution in cosmology.

In addition to the cosmological perturbations in $f(R)$ gravity, even the stability of the GR limit in the more basic cosmic expansion needs to be verified. Furthermore, for any modified gravity theory with higher-order differential equations the stability and the accuracy of the GR limit in the cosmic expansion and in the evolution of the cosmological perturbations must be carefully examined and assessed, especially for the test of modified gravity theories with accurate observational results.

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