NONLINEAR ELECTROMAGNETIC FORCES IN ASTROPHYSICS*

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Electromagnetism becomes a nonlinear theory having (effective) photon–photon interactions due at least to electron–positron fluctuations in the vacuum. We discuss the consequences of the nonlinearity for the force felt by a charge probe particle, and compare the impact of Euler–Kockel QED effective nonlinearity to the possibility of Born–Infeld-type nonlinearity.

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1. Lorentz force and quantum vacuum as polarizable medium

Given the weakness of gravity, a natural question is, if nonlinear electromagnetic interactions can, on stellar scales, lead to transport of matter in ways differing from the Lorentz force expectation. This question is especially pertinent for stellar objects having extreme magnetic fields (magnetars), and we investigate here in a covariant formulation additional forces induced in extreme magnetic fields.

The Lorentz force predicts that a charged particle moving in a magnetic field of arbitrary strength will experience a force normal to both the field direction and the direction of motion. This pivotal property decides the fate of charged particles in the magnetic field-filled Universe. Even a very small field suffices to determine the particle dynamics when acting over a distance large compared to the natural microscopic particle scale, and does so as a function of their velocity. Electric fields play a subdominant role on macroscopic scale since their presence requires a separation or imbalance of charge which normally cannot be maintained for a long time or/and in a large volume.

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The Lorentz force has not been well tested on a macroscopic scale. It could be modified even on planetary scale, considering the material constants of a planetary space plasma, the dielectric polarizibility ϵ and magnetic permeability μ. The question we wish to address here is if, and under what conditions analogue physical effects originate in the empty space vacuum polarizibility due to vacuum fluctuations and/or a natural nonlinearity of electromagnetism. These effects are normally studied in atomic environments, and in comparison to the dominant Coulomb force are exceedingly small. Precision atomic experiments in part compensates this smallness. However, as we will argue here, the astrophysical environment may offer an alternate approach to this physics domain of nonlinear electromagnetism.

The most famous example of modification of the classical electromagnetic theory is the anomalous (since absent unless quantum fluctuations are considered) light–light scattering phenomenon. Early in the development of quantum electrodynamics, Euler and Kockel [1] (and Heisenberg and Euler [2], see below) recognized that electron–positron fluctuations in the vacuum would generate an effective self-interaction for the electromagnetic field. This effect is visualized in the language of Feynman diagrams as the four electromagnetic field legs attached to an electron loop, see Fig. 1.

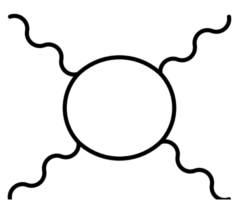

Fig. 1. Feynman diagram expressing effective field–field interaction going through electron–positron fluctuation.

One may also view this phenomenon as the presence of the external field $F^{\mu\nu}$ polarizing the electron fluctuations in the vacuum. This polarization of vacuum fluctuations is analogous to the modification of electromagnetic properties in a material body, *e.g.* plasma, but with several important differences of fundamental origin:

1. The QED vacuum is Lorentz invariant, whereas a material body has a preferred reference frame in which the medium is at rest. This means that, among other things, the framework discussed here and the physics it represents provide a consistent Lorentz invariant theory of nonlinear electromagnetism with sources.

2. In QED, the linear polarization

$$F^{\mu\nu}F_{\mu\nu} \to \left(1 + (\alpha/3\pi)\ln\left(-q^2/m_e^2\right)\right) F^{\mu\nu}F_{\mu\nu}$$

is divergent and is absorbed in the renormalization of charge. In a material body, the linear polarization $\vec{E} \rightarrow \vec{D} = (1 + \overleftrightarrow{\varepsilon}_0)\vec{E}$ is typically finite.

3. The (consequences of QED) polarization we discuss arise from the nonlinear response, such as visualized in Fig. 1, and can be presented in terms of a dielectric tensor $\overleftrightarrow{\varepsilon}$ that is necessarily a covariant function of the applied quasi-constant field.

Heisenberg and Euler showed how to calculate the polarization effect to all orders in the applied (external) field [2]. In so doing, they obtained the first non-perturbative result in quantum field theory, which has been studied and generalized extensively in the intervening years; for a survey of this development we refer the reader to the recent review [3]. The outcome is an effective potential $V_{\rm eff}$ for the electromagnetic field, providing the paradigmatic example of effective field theory, in this case arising by integrating out a "heavy" electron in the low energy ($\omega \ll m_e$) limit of large scale (in space and time) quasi constant fields. Their result contains the Euler and Kockel result seen in Fig. 1 at the lowest finite non-vanishing order (after renormalization) expansion in the fine structure constant α.

However, the full Heisenberg–Euler nonlinearity of electromagnetism is important, considering certain compact stellar objects, 'magnetars', which are now widely expected to be harboring magnetic fields of equal or greater magnitude to the so-called "critical" strength of QED [4, 5]

$$\left|\vec{B}_{\rm c}\right| = \frac{m^2c^2}{e\hbar} = 4.41\ 10^{13}\ \mathrm{G}\,. \tag{1}$$

Near this field strength, all higher order terms in the effective potential become of the same order as is the lowest Euler–Kockel term, see Fig. 1.

We will also explore in this report the effective force in a natural nonlinear extension of Maxwell linear electromagnetism, that is the limiting field Born–Infeld electromagnetism [6]. Considering microscopic physics, we show that new constraints on the Born–Infeld theory require first understanding the dominant QED (Heisenberg–Euler) effects at the percent level.

In the following, we will recall briefly the framework of classical nonlinear electrodynamics and how to derive the dynamics of charged particles and fields. We will discuss why it is necessary to consider energy-momentum conservation explicitly in deriving the effective electromagnetic force and obtain a general formula for the leading terms in the force valid in the leading order in the expansion of the nonlinear effective potential. This corrects our earlier presentation in theoretical detail [7], though the magnitude of the effects remains.

2. Classical dynamics

The Lorentz equation of motion of particles, and Maxwell field equations, are derived by minimizing the total action

$$I_{\rm tot} = \int d^4x \mathcal{L}_{\rm tot}\,, \quad \mathcal{L}_{\rm tot} = -\mathcal{S} + V_{\rm eff} + j^\mu A_\mu + m\tfrac{1}{2}(u^\mu u_\mu - 1)\,. \tag{2}$$

Here the Maxwell electromagnetic action $-\mathcal{S} = (\vec{E}^2 - \vec{B}^2)/2$ is supplemented by an effective potential, such as the Heisenberg–Euler $V_{\rm eff}$. The Maxwell electromagnetic field equations follow from the Euler–Lagrange equations, varying Eq. (2) with respect to A_μ.

Note that the Lorentz force emerges as consequence of variation of the *path* a charged particle takes in the field-filled space. However, a more unified view of paths with fields arises when we consider the Lorentz force as emerging from conservation of energy-momentum tensor. The Feynman path integral formulation of QED offers yet greater unification of both dynamical elements (fields and paths) but this discussion goes well beyond the scope of this report.

At first, we will not specify what $V_{\rm eff}$ is, in order to provide a general derivation and discussion of the consequences of the presence of any $V_{\rm eff}$. The third term in Eq. (2) is the gauge invariant minimal coupling of the electromagnetic potential to the current, and the last term encodes the charged particles' (inertial) mass m and four-velocity $u^\mu = \gamma(1, \vec{v}\,)$. Note that at this point we do not differentiate between electromagnetic and mechanical mass of a particle: there is inertia in the electromagnetic field that accompanies a charged particle and a redefinition of mass is required. This will be achieved below in Eq. (10), where we consider the force related to the nonlinearity of the field.

The effective electromagnetic Lagrangian

$$\mathcal{L} = -\mathcal{S} + V_{\rm eff} \tag{3}$$

can depend only on the scalar and pseudoscalar Lorentz invariants

$$\mathcal{S} = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \qquad \mathcal{P} = \tfrac{1}{4}\widetilde{F}^{\mu\nu}F_{\mu\nu}\,, \qquad \widetilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}\,. \tag{4}$$

Varying Eq. (2) with respect to A_μ then yields

$$\partial_\mu H^{\mu\nu} = j^\nu\,, \tag{5}$$

where $H^{\mu\nu}$ defined as the displacement field tensor (see for example §6 of [8])

$$H^{\mu\nu} \equiv -\frac{\partial\mathcal{L}_{\rm tot}}{\partial F_{\mu\nu}} = F^{\mu\nu} - \frac{\partial V_{\rm eff}}{\partial\mathcal{S}}F^{\mu\nu} - \frac{\partial V_{\rm eff}}{\partial\mathcal{P}}\widetilde{F}^{\mu\nu}\,. \tag{6}$$

When $V_{\rm eff} \to 0$, note that $H^{\mu\nu} \to F^{\mu\nu}$, and Eq. (5) becomes the standard Maxwell equation with source, $\partial_\mu F^{\mu\nu} = j^\nu$.

Current conservation means the field tensor is the 4-dimensional curl of a vector potential A^μ and implies the homogeneous Maxwell equation

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0\,. \tag{7}$$

3. Charged particles in external fields

The dynamics of the charged particle plus field system can be derived from the statement of joint electromagnetic and matter energy-momentum conservation

$$\partial_\mu \left(T^{\mu\nu}_{\text{e.m.}} + T^{\mu\nu}_{\text{matter}}\right) = 0\,. \tag{8}$$

The matter (last) part of the action Eq. (2) defines $T^{\mu\nu}_{\text{matter}}$ with the property

$$\partial_\mu T^{\mu\nu}_{\text{matter}} = u^\mu \partial_\mu (m u^\nu) = dp^\mu / d\tau\,. \tag{9}$$

At this point, one might be tempted to identify $-\partial_\mu T^{\mu\nu}_{\text{e.m.}}$ as the force applied to the charged probe particle. However, this calculation yields the standard Lorentz force $j_\mu F^{\mu\nu}$, even in the presence of nonlinear electromagnetism (see Appendix A). The correct approach to recognize the force is inherent in the work of Born–Infeld [6], which considers the field energy-momentum of individual particles to be identified with their electromagnetic inertia, and hence included in the definition of inertial mass m of the particle sourcing the field. We extend this also to the case of vacuum fluctuation nonlinearity.

By subtracting the energy-momentum of the separate probe particle and external field components as they appear in isolation, only the nonlinear interaction part of the electromagnetic field tensor is retained as sourcing the particle–particle force, see Sec. 4 in Ref. [12]

$$T^{\mu\nu}_{\text{int}} = T^{\mu\nu}_{\text{e.m.}} - T^{\mu\nu}_{\text{p}} - T^{\mu\nu}_{\text{e}}\,. \tag{10}$$

The electromagnetic force is exhibited as the divergence of interaction energy-momentum $T^{\mu\nu}_{\text{int}}$. This procedure will show the usual Lorentz force, as well as further contributions due to the nonlinear interaction of the charged particle's electromagnetic field with the external field, see Appendix B.

For a general nonlinear electromagnetic theory, the energy-momentum tensor can be written [9] (see also Eq. (5) of [10])

$$T^{\mu\nu}_{\text{e.m.}} = \varepsilon T^{\mu\nu}_{\text{Max}} + g^{\mu\nu}\mathcal{T}/4\,, \qquad T^{\mu\nu}_{\text{Max}} = g^{\mu\nu}\mathcal{S} - F^{\mu\kappa}F^{\nu}{}_{\kappa}\,. \tag{11}$$

The deviation from the Maxwell energy-momentum tensor $T^{\mu\nu}_{\rm Max}$ is described by the two Lorentz-scalar functions

$$\varepsilon \;=\; -\frac{\partial(-\mathcal{S}+V_{\rm eff})}{\partial\mathcal{S}} = 1-\frac{\partial V_{\rm eff}}{\partial\mathcal{S}}\,, \tag{12}$$

$$\mathcal{T} \;\equiv\; T^{\mu}_{\mu} = -4\left(V_{\rm eff}-\mathcal{S}\frac{\partial V_{\rm eff}}{\partial\mathcal{S}}-\mathcal{P}\frac{\partial V_{\rm eff}}{\partial\mathcal{P}}\right)\,. \tag{13}$$

The energy-momentum trace $\mathcal{T}$ must arise from $V_{\rm eff}$, because the energy-momentum tensor of the Maxwell theory $T^{\mu\nu}_{\rm Max}$ is traceless, and it must start at order e^4, being related to the presence of (effective) field–field interactions [11], such as displayed in Fig. 1.

For the case of an external magnetic field, the interaction energy-momentum $T^{\mu\nu}_{\rm int}$ is calculated to leading order in Appendix B. This depends on three powers of the external field and is order e^4, which one can see in Fig. 1 by counting the vertices, each of which comes with a power of e[1]. In the case of QED nonlinearity, we are thus finding the effect represented schematically in Fig. 2.

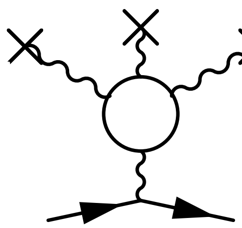

Fig. 2. Diagram expressing contribution to the effective force arising from light-by-light scattering in QED. In the Born–Infeld theory, the loop is to be seen as collapsed to a point.

Taking the divergence of Eq. (B.7), we obtain

$$\frac{dp^{\mu}}{d\tau} = -\partial_{\nu}T^{\mu\nu}_{\rm int} = -j_{\nu}F^{\nu\mu}_{\rm e}+\delta f^{\mu}\,. \tag{14}$$

For the second equality, the Lorentz force (density) is obtained from the divergence of the Maxwell–Lorentz interaction $T^{\mu\nu}_{\rm ep}$ Eq. (B.8) and separated.

[1] We count powers with the QED Heisenberg–Euler $V_{\rm eff}$ in mind, and we will see that the count is simply decreased by 2 for Born–Infeld theory [6] due to the absence of α/π arising from the fermion loop.

The remainder is the sought-for modification

$$\delta f^{\mu} \simeq \frac{\partial\varepsilon}{\partial\mathcal{S}} F^{\mathrm{e}}_{\kappa\lambda} H^{\kappa\lambda}_{\mathrm{p}} \partial_\nu T^{\mu\nu}_{\mathrm{Max,e}} + T^{\mu\nu}_{\mathrm{Max,e}} \partial_\nu \left(\frac{\partial\varepsilon}{\partial\mathcal{S}} F^{\mathrm{e}}_{\kappa\lambda} H^{\kappa\lambda}_{\mathrm{p}}\right) - g^{\mu\nu}\partial_\nu \left(\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}} F^{\mathrm{e}}_{\kappa\lambda} H^{\kappa\lambda}_{\mathrm{p}}\right), \quad (15)$$

where the Maxwell energy-momentum tensor $T^{\mu\nu}_{\mathrm{Max,e}}$ and partial derivatives $\partial\varepsilon/\partial\mathcal{S}, \partial\mathcal{T}/\partial\mathcal{S}$ are evaluated at the external field.

4. Born–Infeld — Euler–Kockel comparison

The action of Born–Infeld electrodynamics is

$$\mathcal{L}_{\mathrm{BI}} = (\mathcal{E}_c)^2 - (\mathcal{E}_c)^2\sqrt{1 + 2\mathcal{S}/\mathcal{E}_c^2 - \mathcal{P}^2/\mathcal{E}_c^4} = -\mathcal{S} + \frac{1}{2\mathcal{E}_c}\left(\mathcal{S}^2 + \mathcal{P}^2\right) - \frac{\mathcal{S}}{2\mathcal{E}_c^2}\left(\mathcal{S}^2 + \mathcal{P}^2\right) + \dots . \quad (16)$$

Born selected originally the limiting electric field strength, $\mathcal{E}_c$, such that the inertia of the electromagnetic field of an electron was equal to the electron mass. Appearance of other fundamental charged particles, the quantum-field theory path to mass renormalization, and further mechanisms (such as Higgs) to generate mechanical particle mass reveal that $\mathcal{E}_c \equiv M^2/e$ is a free parameter, constrained by experiment [13]. Limitations of the possible range of values of the associated mass scale M will be discussed below in comparison with QED.

The expression for the 4-force Eq. (15) is given in terms of ε and $\mathcal{T}$, which we can obtain analytically from Eq. (16). The coefficient functions appearing in Eq. (22) are

$$\frac{\partial\varepsilon}{\partial\mathcal{S}} = -\left(1 + 2\mathcal{S}/\mathcal{E}_c^2 - \mathcal{P}^2/\mathcal{E}_c^4\right)^{-3/2}\mathcal{E}_c^{-2} = -\varepsilon^3\mathcal{E}_c^{-2} \simeq -\frac{e^2}{M^4} + \frac{3e^4\mathcal{S}}{M^8} - \frac{e^6}{M^{12}}\frac{3}{2}\left(5\mathcal{S}^2 + \mathcal{P}^2\right) + \dots, \quad (17)$$

$$\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}} = \varepsilon^3\left(\mathcal{S}/\mathcal{E}_c^2 - \mathcal{P}^2/\mathcal{E}_c^4\right) \simeq \frac{\mathcal{S}}{\mathcal{E}_c^2} - 3\frac{\mathcal{S}^2}{\mathcal{E}_c^4} - \frac{\mathcal{P}^2}{\mathcal{E}_c^4}\dots \quad (18)$$

given also in weak-field expansion in the second line.

The Euler–Kockel effective potential, which is the first term of Euler–Heisenberg power series in e^2/m_e^4 is given by

$$V_{\mathrm{eff}} \simeq \frac{\alpha}{90\pi}\frac{e^2}{m_e^4}\left(4\mathcal{S}^2 + 7\mathcal{P}^2\right) + \frac{2\alpha}{315\pi}\frac{e^4}{m_e^8}\mathcal{S}\left(8\mathcal{S}^2 + 13\mathcal{P}^2\right) + \dots . \quad (19)$$

The common single power of α/π is due to the calculation being a one-loop evaluation of electron fluctuations. For the coefficient functions, one finds

$$\frac{\partial\varepsilon}{\partial\mathcal{S}} \simeq -\frac{4}{45}\frac{\alpha}{\pi}\frac{e^2}{m_e^4} + \frac{32}{105}\frac{\alpha}{\pi}\frac{e^4}{m_e^8}\mathcal{S} + \dots\,, \tag{20}$$

$$\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}} \simeq \frac{4}{45}\frac{\alpha}{\pi}\frac{e^2}{m_e^4}\mathcal{S} - \frac{4}{315}\frac{\alpha}{\pi}\frac{e^4}{m_e^8}\left(24\mathcal{S}^2 + 13\mathcal{P}^2\right) + \dots\,. \tag{21}$$

Let us compare Born–Infeld results Eq. (17) and Eq. (18) to Euler–Kockel result Eq. (20) and Eq. (21): we see that the QED effects are apparently more strongly suppressed because they arise from quantum corrections, signaled by the presence of α/π. With the smallness of α/π, the Born–Infeld coefficients are $\sim 5000(m_e^4/M^4)$ times the QED coefficients and clearly this is excluded by precision QED tests. In fact, the latest published constraint on Born–Infeld $M \gtrsim 60$ MeV [14], meaning $m_e^4/M^4 \lesssim 6\,10^{-5}$, which assures that QED dominates Born–Infeld corrections.

5. Physical effects of the nonlinear force

The force derived from the nonlinear action of QED or/and Born–Infeld type theory will originate in an increase in field energy when superposing fields. Thus in general, the force should act in such a way as to screen and reduce the total field strength. In the case of a magnetic-only external field, this means the sign of the force depends on $\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}$ displaying the relative orientation of the external field $\vec{B}_\mathrm{e}$ and probe magnetic field $\vec{H}_\mathrm{p}$. For example, when the fields are aligned, $\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p} > 0$ and the force acts to expel the probe from this domain. On the other hand, when the fields are anti-aligned, the force acts to pull probe deeper into the strong field domain to increase the screening.

The effect is manifested in the 4th (time-like) component of the force determines the change in energy of the probe current and is related the 3-force given shortly by the covariant requirement $u_\mu dp^\mu/d\tau = 0$. In the 0-component of Eq. (15), we take the external electric field to be negligible, so the first term in Eq. (15) does not appear. With $F^\mathrm{e}_{\kappa\lambda}H^{\kappa\lambda}_\mathrm{p} = 2\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}$, we obtain

$$\begin{aligned}\delta f^0 &= \mathcal{S}_\mathrm{e}\,\partial_t\left(\frac{\partial\varepsilon}{\partial\mathcal{S}}2\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}\right) - \partial_t\left(\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}}2\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}\right)\\ &= 4\mathcal{S}_\mathrm{e}\partial_t\left(\frac{\partial\varepsilon}{\partial\mathcal{S}}\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}\right) + 2\frac{\partial\varepsilon}{\partial\mathcal{S}}\vec{B}_\mathrm{e}\cdot\vec{H}_\mathrm{p}\partial_t\mathcal{S}_\mathrm{e}\,.\end{aligned} \tag{22}$$

To obtain the second line, we use the identity

$$-\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}} = \mathcal{S}\frac{\partial\varepsilon}{\partial\mathcal{S}} + \mathcal{P}\frac{\partial\varepsilon}{\partial\mathcal{P}} \tag{23}$$

which follows from the respective definitions Eqs. (12) and (13). Again, since the external electric field is vanishing, the second term in this identity proportional to $\mathcal{P} = \vec{E} \cdot \vec{B}$ does not appear.

The 3-force corresponding to Eq. (22) is

$$\begin{aligned}
\delta\vec{f} &= 2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{\partial\varepsilon}{\partial\mathcal{S}} \vec{B}_\mathrm{e} \times \left(\vec{\nabla} \times \vec{B}_\mathrm{e}\right) \\
&\quad + \left(\left|\vec{B}_\mathrm{e}\right|^2 \vec{\nabla} - \vec{B}_\mathrm{e}\left(\vec{B}_\mathrm{e} \cdot \vec{\nabla}\right)\right)\left(2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{\partial\varepsilon}{\partial\mathcal{S}}\right) + \vec{\nabla}\left(2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}}\right) \\
&= 2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{\partial\varepsilon}{\partial\mathcal{S}} \vec{B}_\mathrm{e} \times \left(\vec{\nabla} \times \vec{B}_\mathrm{e}\right) - 2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{\partial\varepsilon}{\partial\mathcal{S}} \vec{\nabla}\mathcal{S}_\mathrm{e} \\
&\quad + \left(\frac{\left|\vec{B}_\mathrm{e}\right|^2}{2} \vec{\nabla} - \vec{B}_\mathrm{e}\left(\vec{B}_\mathrm{e} \cdot \vec{\nabla}\right)\right)\left(2\vec{B}_\mathrm{e} \cdot \vec{H}_\mathrm{p} \frac{\partial\varepsilon}{\partial\mathcal{S}}\right) .
\end{aligned} \tag{24}$$

The second equality is obtained again using Eq. (23). The first term in Eq. (24) corresponds to the first term in Eq. (15) and as seen in consideration of δf^0 does no work on the probe current. It remains to be shown that the remaining two terms in Eq. (24) are physically relevant.

We consider two competing forces in an astrophysical plasma near a compact star: the gravity of the star and the internal energy (pressure) of the plasma. For order of magnitude estimates of the force Eq. (24), we can use the Newtonian approximation for the star's gravity and replace the gradients by $1/L$, where L is the compact star length scale, *i.e.* its radius ~ 10 km. Then, the ratio of radial forces is

$$\frac{\left|\hat{r} \cdot \delta\vec{f}\right|}{\left|\hat{r} \cdot \vec{f}_\mathrm{grav}\right|} \simeq \frac{\partial\varepsilon}{\partial\mathcal{S}} \frac{\left|\vec{B}_\mathrm{e}\right|^3 \left|\vec{H}_\mathrm{p}\right|}{L} \frac{L^2}{GM\rho} = \frac{4\alpha}{45\pi} \frac{\left|\vec{B}_\mathrm{e}\right|^2}{\left|\vec{B}_\mathrm{c}\right|^2} \frac{\left|\vec{B}_\mathrm{e}\right| \left|\vec{H}_\mathrm{p}\right|}{\rho} \frac{L}{GM} , \tag{25}$$

where ρ is the mass density of the plasma $\rho \sim m_\mathrm{p} 10^{11}$ cm^{-3} for a neutral astrophysical plasma. G is the Newton gravitational constant and M the mass of the star, so that $L/GM \sim 5$–10 for a compact star. We have replaced $\partial\varepsilon/\partial\mathcal{S}$ by its QED value so that $|\vec{B}_\mathrm{c}|$ is the QED "critical" field strength Eq. (1). $|\vec{H}_\mathrm{p}| \gtrsim$ a few mG according to estimates and models of accreting plasmas [15]. Putting in the numbers shows that the nonlinear electromagnetic force dominates gravity for $|\vec{B}_\mathrm{e}|/|\vec{B}_\mathrm{c}| \gtrsim 10^{-7}$, confirming the previous analysis [7]. We estimate the pressure P in the plasma from the ideal gas law and compare it to the interaction energy density, which is essentially the force derived here without the gradients. Thus, the ratio is

$$\frac{\left|T^{00}_{\rm int}\right|}{P} = \frac{\partial\varepsilon}{\partial\mathcal{S}}\frac{\left|\vec{B}_{\rm e}\right|^3\left|\vec{H}_{\rm p}\right|}{nT}\,, \tag{26}$$

where n is the number density ($\sim 10^{11}$ cm^{-3}) and T is the temperature, suggested by observations to be of the order of 10–100 eV [16]. Since this differs from the previous estimate Eq. (25) only by the factors $m_{\rm p}/T \sim 10^7$ and GM/L, the immediate conclusion is the nonlinearity of the electromagnetic interaction may be more important than the plasma dynamics near to the surface of the star. Note, of course, that a dipole field falls off rapidly as $|B_{\rm e}| \sim 1/r^3$ and these nonlinear field effects will be subdominant farther from the surface, say at 200 times the stellar radius where $|\vec{B}_{\rm e}|^3$ has fallen to 10^{-7} its value at the surface.

6. Conclusions

After reviewing the framework of nonlinear electromagnetism, we showed here how to calculate the force arising from the nonlinear field–field interaction, shown schematically in Fig. 2. The procedure involves separately identifying the polarization of energy-momentum tensor related to presence of two components, here external field and particle field, so that we can account for the electromagnetic mass of particles.

With only magnetic fields and Lorentz–Maxwell electrodynamics, the energy of charged particles near stellar objects is determined solely by gravity. We have shown that once Euler and Kockel [1] (effective) QED nonlinear electromagnetism is accounted for, magnetic fields can do work on charged particles and this effect beats out gravity for fields that are quite strong yet still very far from critical. This insight may have considerable physics impact, considering that there is no *a priori* limit to the magnetic field that a ferromagnetic star can source. It is for this reason that exploration of the physical consequences arising in ultra strong magnetic fields remains a topic of current intense discussion relevant to extreme astrophysical events [17] and in various areas of elementary matter physics [18–23].

To close, we note that the current report must be seen as a first step on the way to understand the classical charged particle dynamics in presence of ultra strong external magnetic fields and gravity. Beyond the effects we considered here, we further expect radiation reaction to be relevant in proximity of the critical field strength where acceleration is so strong that the radiation field impacts the source dynamics.

Appendix A

Finding the Lorentz force in NLEM

In this appendix, we calculate the divergence of the electromagnetic energy-momentum tensor Eq. (11)

$$\partial_\mu T^{\mu\nu}_{\text{e.m.}} = (\partial_\mu \varepsilon) T^{\mu\nu}_{\text{Max}} + \varepsilon \partial_\mu T^{\mu\nu}_{\text{Max}} + \partial_\mu g^{\mu\nu} \mathcal{T}/4 \,. \tag{A.1}$$

First, we write out the first two terms

$$(\partial_\mu \varepsilon) = \left(-\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \right) (g^{\mu\nu} \mathcal{S} - F^{\mu\kappa} F^{\nu}{}_{\kappa}) \,, \tag{A.2}$$

$$\partial_\mu T^{\mu\nu}_{\text{Max}} = \partial_\mu (g^{\mu\nu} \mathcal{S} - F^{\mu\kappa} F^{\nu}{}_{\kappa}) \tag{A.3}$$

with $\mathcal{L} = -\mathcal{S} + V_{\text{eff}}$ a shorthand for the total electromagnetic Lagrangian. Then for the divergence of the trace $\mathcal{T}$, we use the form given in Eq. (13)

$$\partial_\mu \mathcal{T}/4 = + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \right) \mathcal{S} + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{P}} \right) \mathcal{P} \tag{A.4}$$

simplified by virtue of $d\mathcal{L} = (\partial \mathcal{L}/\partial \mathcal{S}) d\mathcal{S} + (\partial \mathcal{L}/\partial \mathcal{P}) d\mathcal{P}$. Note that it does not matter whether one writes $\mathcal{L}$ or V_{eff} in the variation because any terms linear in $\mathcal{S}$ manifestly cancel between the first and second terms of Eq. (13). The first term in Eq. (A.4) then cancels with the first term in Eq. (A.2). Next, we observe that

$$-g^{\mu\nu} \mathcal{P} = F^{\mu\kappa} \widetilde{F}_{\kappa}{}^{\nu} \tag{A.5}$$

and find for Eq. (A.1)

$$\begin{aligned} \partial_\mu T^{\mu\nu}_{\text{e.m.}} &= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \right) F^{\mu\kappa} F^{\nu}{}_{\kappa} - \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \partial_\mu (g^{\mu\nu} \mathcal{S} - F^{\mu\kappa} F^{\nu}{}_{\kappa}) + g^{\mu\nu} \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{P}} \right) \mathcal{P} \\ &= \left(\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \right) F^{\mu\kappa} + \frac{\partial \mathcal{L}}{\partial \mathcal{S}} \partial_\mu F^{\mu\kappa} + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \mathcal{P}} \right) \widetilde{F}^{\mu\kappa} \right) F^{\nu}{}_{\kappa} \\ &\quad + \frac{\partial \mathcal{L}}{\partial \mathcal{S}} (F^{\mu\kappa} \partial_\mu F^{\nu}{}_{\kappa} - g^{\mu\nu} \partial_\mu \mathcal{S}) \,. \end{aligned} \tag{A.6}$$

Now using the definition of the displacement tensor Eq. (6), we recognize the first three terms in brackets as an expanded expression of the divergence of $-H^{\mu\nu}$. Note that $\partial_\mu \widetilde{F}^{\mu\nu}$ vanishes according to the homogeneous Maxwell equation (7). From Eq. (7) one can also show that

$$\partial_\mu \mathcal{S} = \tfrac{1}{2} F^{\kappa\lambda} \partial_\mu F_{\kappa\lambda} = -\tfrac{1}{2} F^{\kappa\lambda} (\partial_\kappa F_{\lambda\mu} + \partial_\lambda F_{\mu\kappa}) = -F^{\kappa\lambda} \partial_\kappa F_{\lambda\mu} \tag{A.7}$$

which means the final two terms in Eq. (A.6) cancel. Thus, we obtain

$$\partial_\mu T^{\mu\nu}_{\text{e.m.}} = j_\mu F^{\mu\nu} \,. \tag{A.8}$$

A slightly different proof, starting from a different but equivalent form of $T^{\mu\nu}_{\text{e.m.}}$ can be found in §6 of [8].

Appendix B

Lorentz force correction derivation

Considering the external field to provide the dominant part of the electromagnetic energy-momentum, we expand in the displacement tensor of the probe particle

$$T^{\mu\nu}_{\rm int} = T^{\mu\nu}_{\rm e.m.} - T^{\mu\nu}_{\rm e} - T^{\mu\nu}_{\rm p} = T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)}\,, \tag{B.1}$$

$$T^{\mu\nu}_{(1)} = \left.\frac{\partial T^{\mu\nu}_{\rm e.m.}}{\partial H^{\alpha\beta}}\right|_{\rm e} H^{\alpha\beta}_{\rm p}\,, \tag{B.2a}$$

$$T^{\mu\nu}_{(2)} = \left.\frac{\partial^2 T^{\mu\nu}_{\rm e.m.}}{\partial H^{\alpha\beta}\partial H^{\kappa\lambda}}\right|_{\rm e} \frac{H^{\kappa\lambda}_{\rm p} H^{\alpha\beta}_{\rm p}}{2} - T^{\mu\nu}_{\rm p} \tag{B.2b}$$

with the subscript "e" reminding that derivatives are evaluated for the external field. The zeroth order term is just $T^{\mu\nu}_{\rm e}$, which is subtracted, and the energy-momentum of the probe particle is found at second order in $T^{\mu\nu}_{(2)}$ and subtracted. The expansion is in terms of the probe particle displacement tensor, because it is defined by the Maxwell equation with source Eq. (5).

The derivatives with respect to $H^{\mu\nu}$ are related to derivatives with respect to $F^{\mu\nu}$ by

$$\frac{\partial F^{\alpha\beta}}{\partial H^{\mu\nu}} = \tfrac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right) + \tfrac{\partial^2 V_{\rm eff}}{\partial F_{\mu\nu}\partial F_{\alpha\beta}} + \ldots \tag{B.3}$$

obtained from inverting Eq. (6). The 1/2 is a symmetry factor arising from the equivalence under permutations of antisymmetric indices. The two terms shown correspond to expanding up to order e^4 as needed in the present approximation. Since $T^{\mu\nu}$ is quadratic in $F^{\mu\nu}$, expanding Eq. (B.3) in powers of e is the primary expansion involved in obtaining the result here.

Using Eq. (11), the calculation of the tensor derivatives in Eq. (B.2) is split into pieces:

$$\frac{\partial T^{\mu\nu}_{\rm Max}}{\partial H^{\kappa\lambda}} = \frac{\partial T^{\mu\nu}_{\rm Max}}{\partial F^{\rho\sigma}}\frac{\partial F^{\rho\sigma}}{\partial H^{\kappa\lambda}} = \frac{\partial T^{\mu\nu}_{\rm Max}}{\partial F^{\rho\sigma}}\left(\tfrac{1}{2}(\delta^\rho_\kappa\delta^\sigma_\lambda - \delta^\rho_\lambda\delta^\rho_\kappa) + \tfrac{\partial^2 V_{\rm eff}}{\partial F^{\rho\sigma}\partial F_{\kappa\lambda}}\right)\,, \tag{B.4}$$

$$\frac{\partial \varepsilon}{\partial H^{\kappa\lambda}} = \frac{\partial \varepsilon}{\partial F^{\rho\sigma}}\frac{\partial F^{\rho\sigma}}{\partial H^{\kappa\lambda}}\,, \qquad \frac{\partial \mathcal{T}}{\partial H^{\kappa\lambda}} = \frac{\partial \mathcal{T}}{\partial F^{\rho\sigma}}\frac{\partial F^{\rho\sigma}}{\partial H^{\kappa\lambda}}\,. \tag{B.5}$$

Because $V_{\rm eff}$ and so $\varepsilon, \mathcal{T}$ depend only on $\mathcal{S}, \mathcal{P}$, derivatives with respect to $F^{\mu\nu}$ can be split into partial derivatives

$$\frac{\partial\,\cdot}{\partial F^{\mu\nu}} = \frac{\partial\,\cdot}{\partial \mathcal{S}} F_{\mu\nu} + \frac{\partial\,\cdot}{\partial \mathcal{P}}\widetilde{F}_{\mu\nu}\,. \tag{B.6}$$

We see then that $\partial\varepsilon/\partial\mathcal{S}, \partial\mathcal{T}/\partial\mathcal{S}, \partial\varepsilon/\partial\mathcal{P}, \partial\mathcal{T}/\partial\mathcal{P}$ are all of the order of e^4, and the next-to-leading-order term in $\partial F/\partial H$ Eq. (B.3) can be dropped.

Looking to the study of astrophysical magnetic fields, we specialize to the case $\vec{E}\cdot\vec{B} = \mathcal{P} = 0$ for the external field. Because $V_{\rm eff}$ must be parity invariant, it contains only even powers of $\mathcal{P}^2$ and, therefore, the partial derivatives $\partial\varepsilon/\partial\mathcal{P}$ and $\partial\mathcal{T}/\partial\mathcal{P}$ vanish being proportional to $\mathcal{P}$. In this case then

$$T^{\mu\nu}_{\rm int} = T^{\mu\nu}_{\rm ep} - T^{\mu\nu}_{\rm Max,e}\frac{\partial\varepsilon}{\partial\mathcal{S}}F^{\rm e}_{\alpha\beta}H^{\alpha\beta}_{\rm p} + g^{\mu\nu}\frac{1}{4}\frac{\partial\mathcal{T}}{\partial\mathcal{S}}F^{\rm e}_{\alpha\beta}H^{\alpha\beta}_{\rm p}\,. \tag{B.7}$$

Here $T^{\mu\nu}_{\rm Max,e}$ is the Maxwell energy-momentum tensor, compare Eq. (11), for the external field and the (linear) Maxwell–Lorentz interaction is separated

$$T^{\mu\nu}_{\rm ep} = -\left(F^{\mu\kappa}_{\rm e}H^{\nu\lambda}_{\rm p} + F^{\nu\lambda}_{\rm e}H^{\mu\kappa}_{\rm p}\right)g_{\kappa\lambda} + g^{\mu\nu}\tfrac{1}{2}F^{\rm e}_{\alpha\beta}H^{\alpha\beta}_{\rm p}\,. \tag{B.8}$$

By going to the rest frame of the probe particle, it is easy to show that this term produces the Lorentz force

$$\partial_\mu T^{\mu\nu}_{\rm ep} = j_\mu F^{\mu\nu}_{\rm e}\,. \tag{B.9}$$

Taking the negative divergence of Eq. (B.7) yields Eq. (15).

Now, studying the second order term $T^{\mu\nu}_{(2)}$ it turns out there are terms of the order of e^4. Most can be seen to have form of the electromagnetic energy-momentum of the probe $T^{\mu\nu}_{\rm p}$ (in its nonlinear form Eq. (11)). One term has the form of a scalar modification to the Lorentz force, which is seen by differentiating ε and $T^{\mu\nu}_{\rm Max}$ each once. However, we will consider the probe field to be much weaker than the external field, in which case these terms are all smaller than those calculated by a factor $|\vec{H}_{\rm p}|/|\vec{B}_{\rm e}| \ll 1$. Incorporating these terms would provide the complete result for the 4-force at order e^4.

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REFERENCES

[1] H. Euler, B. Kockel, *Naturwiss.* **23**, 246 (1935).

[2] W. Heisenberg, H. Euler, *Z. Phys.* **98**, 714 (1936).

[3] G.V. Dunne, `arXiv:hep-th/0406216`.

[4] J.H. Taylor, R.N. Manchester, A.G. Lyne, *Astrophys. J. Suppl.* **88**, 529 (1993).

[5] S. Mereghetti, *Astron. Astrophys. Rev.* **15**, 225 (2008).

[6] M. Born, L. Infeld, *Proc. R. Soc. Lond.* **A134**, 425 (1934).

[7] L. Labun, J. Rafelski, *Phys. Lett.* **B687**, 133 (2010) [arXiv:0810.1323 [hep-ph]].

[8] I. Bialynicki-Birula, Z. Bialynicka-Birula, *Quantum Electrodynamics*, Pergamon, Oxford 1975, p. 548.

[9] J.S. Schwinger, *Phys. Rev.* **82**, 664 (1951).

[10] L. Labun, J. Rafelski, *Phys. Rev.* **D81**, 065026 (2010).

[11] S.L. Adler, J.C. Collins, A. Duncan, *Phys. Rev.* **D15**, 1712 (1977).

[12] J. Rafelski, W. Greiner, L.P. Fulcher, *Nuovo Cim.* **B13**, 135 (1973).

[13] J. Rafelski, L.P. Fulcher, W. Greiner, *Phys. Rev. Lett.* **27**, 958 (1971).

[14] G. Soff, J. Rafelski, W. Greiner, *Phys. Rev.* **A7**, 903 (1973).

[15] M. Pessah, D. Psaltis, *Astrophys. J* **628**, 879 (2005).

[16] J.C.A. van Peet *et al.*, *Astron. Astrophys.* **497**, 805 (2009).

[17] R.C. Duncan, C. Thompson, *Astrophys. J. Lett.* **392**, L9 (1992); C. Thompson, R.C. Duncan, *Mon. Not. R. Astron. Soc.* **275**, 255 (1995); *Astrophys. J.* **473**, 322 (1996).

[18] A.K. Harding, D. Lai, *Rep. Prog. Phys.* **69**, 2631 (2006) [arXiv:astro-ph/0606674].

[19] A. Colaiuda, V. Ferrari, L. Gualtieri, J.A. Pons, *Mon. Not. R. Astron. Soc.* **385**, 2080 (2008) [arXiv:0712.2162 [astro-ph]].

[20] A. Rabhi, H. Pais, P.K. Panda, C. Providencia, *J. Phys. G* **36**, 115204 (2009) [arXiv:0909.1114 [nucl-th]].

[21] E.J. Ferrer, V. de la Incera, C. Manuel, *Phys. Rev. Lett.* **95**, 152002 (2005) [arXiv:hep-ph/0503162].

[22] E.J. Ferrer *et al.*, *Phys. Rev.* **C82**, 065802 (2010) [arXiv:1009.3521 [hep-ph]].

[23] M. Sinha, B. Mukhopadhyay, A. Sedrakian, arXiv:1209.5611 [astro-ph.HE].