Shear Viscosity of a Non-Relativistic Conformal Gas in Two Dimensions

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The shear viscosity, η, of a fermi gas with non-relativistic conformal symmetry in two spatial dimensions is investigated. We find that η/s, s being the entropy density, diverges as a gas of free particles in this system. This is in contrast to the $\eta/s = 1/4\pi$ found using the non-relativistic AdS/CFT correspondence, which requires a strongly interacting CFT. It implies that the unitary fermi gas in two spatial dimensions is not likely to have a weakly interacting gravity dual.

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I. MOTIVATION

Recently the AdS/CFT correspondence, originally proposed for supersymmetric conformal field theories [1–3] , has been conjectured to exist in non-relativistic conformal field theories (NRCFT) [4–8] . One of the goals is to apply the tool to cold atomic systems in the unitarity limit, where the two-body S-wave scattering length diverges (or the two-body binding energy B vanishes) and the system of two-component fermions exhibits a non-relativistic conformal symmetry [9]* . Later, the NR AdS/CFT correspondence was generalized to finite temperature [12–14]. In particular, a special kind of black hole solution of type IIB was constructed using the Null Melvin Twist technique. The theory was identified as the gravity dual of a $d = 2$ NRCFT, d being the number of spatial dimensions, at finite density and finite temperature. The resulting shear viscosity (η) to entropy density (s) ratio, η/s, is identical to $1/4\pi$ as in the relativistic cases using the AdS/CFT correspondence [15–17].

However, it is known that in $d = 1$ and 2 an attractive contact interaction between two fermions will always give rise to a bound state. Thus, zero binding energy implies a free system [18–21]. In this paper, we demonstrate this known result in the effective field theory (EFT) language. We conclude that $\eta/s \to \infty$ in $d = 2$ when $B = 0$. This implies that the unitary fermi gas in $d = 2$ is not likely to have a weakly interacting gravity dual. It will be interesting to find some strongly interacting NRCFT candidates in $d = 2$ that

[*] If the system is bosonic or is fermionic but with more than two components (e.g., with 2 spin and 2 isospin states), then the three-body interaction can generate a scale to break the conformal symmetry [10, 11]. For a system with two component (spin up and down) fermions, the three-body interaction is derivatively coupled and is of higher order by the Pauli exclusion principle.

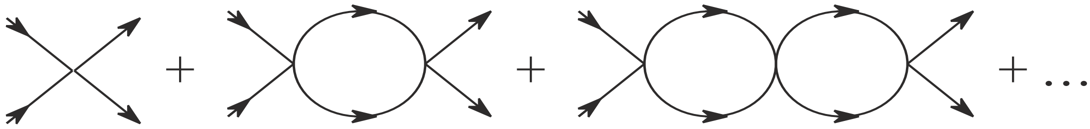

FIG. 1: Leading order diagrams for two fermion scattering.

might exhibit the NR AdS/CFT correspondence.

II. THE FIELD THEORY APPROACH

For convenience, we use the EFT approach to compute the two-body scattering amplitudes in various dimensions. This approach is equivalent to solving the Schrödinger equation with a delta function potential. One can use a square well potential to solve the Schroedinger equation then send the width of the potential to zero such that the width does not break the conformal invariance.

The leading order EFT Lagrangian in the energy expansion for two-component, non-relativistic fermions is [22–24]

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) \psi - C_0 \left(\psi^\dagger \psi \right)^2 , \tag{1}$$

where four fermion contact interactions with derivatives are higher order and are neglected. There is no particle pair creation in a non-relativistic theory, so there is no contribution from the "eye diagrams." The leading order two-body interaction through the bubble diagrams shown in Fig. 1 gives rise to the scattering amplitude

$$i\mathcal{A} = -i \frac{C_0}{1 - C_0 I} = -i \frac{1}{1/C_0 - I} , \tag{2}$$

where I denotes the loop integral. In the center-of-mass (CM) frame, the system has energy E and in dimensional regularization

$$\begin{aligned} I &= -i \left(\frac{\mu}{2} \right)^{d-D} \int \frac{d^{D+1}q}{(2\pi)^{D+1}} \left(\frac{i}{\frac{E}{2} + q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon} \right) \left(\frac{i}{\frac{E}{2} - q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon} \right) \\ &= \left(\frac{\mu}{2} \right)^{d-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{E - \frac{\mathbf{q}^2}{M} + i\epsilon} \\ &= -M \left(-ME - i\epsilon \right)^{\frac{D-2}{2}} \Gamma \left(\frac{D-2}{2} \right) \left(\frac{\mu}{2} \right)^{d-D} (4\pi)^{\frac{-D}{2}} , \end{aligned} \tag{3}$$

where d is the number of spatial dimensions and D will be expanded around d. If the interaction generates a bound state with bounding energy B, then $\mathcal{A}$ will have a pole at $E = -B$. Using Eqs. (2) and (3), this implies

$$\frac{1}{C_0} = -M\left(MB\right)^{\frac{D-2}{2}}\Gamma\left(\frac{D-2}{2}\right)\left(\frac{\mu}{2}\right)^{d-D}\left(4\pi\right)^{\frac{-D}{2}}. \tag{4}$$

We will be interested in the cases with $d = 1, 2, 3$.

$$\frac{1}{C_0} = \begin{cases} \dfrac{M\sqrt{MB}}{4\pi}\,,\ d=3 \\ \dfrac{M}{2\pi\left(D-2\right)} + \frac{M}{4\pi}\left(\ln\left[\dfrac{MB}{\mu^2}\right] + \gamma_E\right),\ d=2 \\ -\frac{1}{2}\sqrt{\dfrac{M}{B}}\,,\ d=1. \end{cases} \tag{5}$$

When the system is tuned to have a bound state with zero binding energy ($B = 0$), we see that in $d = 3$, $C_0 \to \infty$, which is corresponding to the unitarity limit where the two-body scattering length is infinite. However, in $d = 1$, $C_0 \to 0$, which is the free case. In $d = 2$, $1/C_0$ has to absorb the $1/\left(D-2\right)$ pole and it does not directly reflect the strength of the coupling. However, we can analyze the scattering amplitude,

$$\mathcal{A} = \begin{cases} -\dfrac{4\pi}{M\sqrt{M}}\dfrac{1}{\sqrt{B}-i\sqrt{E}}\,,\ d=3 \\ -\dfrac{4\pi}{M}\dfrac{1}{\left(\ln\left[\frac{B}{E}\right]+i\frac{\pi}{2}\right)}\,,\ d=2 \\ 2\sqrt{\dfrac{B}{M}}\dfrac{i\sqrt{E}}{\sqrt{B}+i\sqrt{E}}\,,\ d=1. \end{cases} \tag{6}$$

By design, the amplitude $\mathcal{A}$ has a pole at $E = -B$ (the correct limit for $B \to 0$ is to take $E = -B$ first then take $B \to 0$). We see that for $B = 0$ and $E > 0$ particles do not interact ($\mathcal{A} = 0$) in both $d = 1$ or $d = 2$. The same conclusion was obtained in [18, 19] by solving the Schroedinger equation.

The above analysis implies that the shear viscosity $\eta \to \infty$ when $d = 2$ (while η is not defined in $d = 1$). Note that the pole in the two-particle scattering amplitude at $E = 0$ has no effect on η. This is because η reflects the time needed for a system to relax to thermal equilibrium once it is perturbed away from equilibrium. However, $E = 0$ in the CM frame means there is no relative momentum between particles scattering in any inertia frame. So there is no momentum rearrangement and no relaxation to thermal equilibrium during the scattering. Thus, as far as computing η is concerned, the system is a free system and $\eta \to \infty$. Since the entropy density s is finite for a free system, $\eta/s \to \infty$ for $d = 2$ when $B = 0$.

III. GRAVITATIONAL ASPECT

In the gravity side, one might wonder if η/s could have different values for the free fermion limit and the $B = 0$ limit. Just as in $d = 3$, both limits satisfy the same NR conformal symmetry but different boundary conditions [4]. In the following, however, we argue that in $d = 2$, the two limits degenerate to the free fermion limit.

Let us recall the operator-field correspondence in NR AdS/CFT. We consider a minimally coupled massive scalar field ϕ with mass m propagating in the following background of d spatial dimensions, which exhibits a full Schroedinger symmetry [4] (see [25] for an earlier work):

$$ds^2 = -\frac{2(dx^+)^2}{z^4} + \frac{-2dx^+dx^- + dx^i dx^i + dz^2}{z^2}. \tag{7}$$

Here the two null-like Killing directions $\partial/\partial x^+$ and $\partial/\partial x^-$ are associated with the energy ω and mass M of the system, and a discrete mass spectrum can be easily realized by making x^- periodic. Given a plane wave ansatz for a scalar field,

$$\phi(x^+, x^-, x^i, z) = e^{i\omega x^+ + iMx^- + ik_i x^i} u(z), \tag{8}$$

one obtains two independent solutions [4]:

$$u_\pm = z^{d/2+1} K_{\pm\nu}(pz), \qquad p = \sqrt{\vec{k}^2 - 2M\omega}, \qquad \nu = \sqrt{m^2 + 2M^2 + \left(\frac{d+2}{2}\right)^2}. \tag{9}$$

For $0 < \nu < 1$, both solutions are renormalizable, and the corresponding operators have dimensions $\Delta_\pm = d/2 + 1 \pm \nu$. In particular, one is free to choose $\nu = d/2 - 1$ such that the operators have dimensions d and 2, respectively, corresponding to the dimension of the $(\psi\psi)$ operator for free fermions and fermions at unitarity. Note that, for $d = 2 (\nu = 0)$, $u_\pm$ scales like z^2 and $z^2 \ln(z/z_0)$, where z_0 is some scale breaking the conformal invariance [26]. Thus only the z^2 solution is allowed, and we are left with a single picture of free fermions.

IV. CONCLUSIONS

We have shown that for a system of two-component non-relativistic fermions with $d = 2$, as the two-body binding energy B is tuned to be zero, $\eta/s \to \infty$ as a free system. This implies the unitary fermi gas in $d = 2$ is not likely to have a weakly interacting gravity dual. It will be interesting to find some strongly interacting NRCFT candidates in $d = 2$ that might exhibit the NR AdS/CFT correspondence.

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