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DETERMINISM BENEATH COMPOSITE QUANTUM SYSTEMS

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This paper aims at the development of 't Hooft's quantization proposal to describe composite quantum mechanical systems. In particular, we show how 't Hooft's method can be utilized to obtain from two classical Bateman oscillators a composite quantum system corresponding to a quantum isotonic oscillator. For a suitable range of parameters, the composite system can be also interpreted as a particle in an effective magnetic field interacting through a spin-orbital interaction term. In the limit of a large separation from the interaction region we can identify the irreducible subsystems with two independent quantum oscillators.

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1. Introduction

Almost two decades ago 't Hooft outlined a program that strived to provide an explanatory (as opposed to merely predictive) basis for quantum mechanics (QM).[1] The program itself has been variably called "pre-quantization",[2] "emerged quantum mechanics",[3] "'t Hooft's deterministic quantum theory" [4] or "'t Hooft's quantization proposal".[5]

The key observation of 't Hooft is that there exists a simple family of deterministic (though non-standard) dynamical systems that can be described by means

of Hilbert space techniques without loosing their deterministic character. Only after enforcing certain constraints expressing information loss, one obtains genuine quantum systems. In this picture, the enormous amount of information that would be lost in the process of "coarse graining" from Planckian-scale – 10^{19}GeV, to our current observational scale – 10^3GeV, would lead to formation of equivalence classes, in the sense that many distinct states of the Planckian-scale dynamics would go into a single state of the observational scale dynamics. Dynamical degrees of freedom on the primordial Planck scale are also known as *be-ables* — as oppose to *observables*, i.e., degrees of freedom that are affiliated with the observational energy scale.

't Hooft himself provided explicit examples of be-able systems that give rise, at a macroscopic level, to a genuine quantum behavior.[1,6] This line of thoughts has been further progressed, e.g., in Refs. 7–12. For consistency of the pre-quantization scheme it is, however, important to show that, whenever an interacting composite QM system is obtained from a be-able dynamics, then also its identifiable subparts (i.e., subsystems) are fully quantal. This in particular means that both the composite QM system and its identifiable subsystems must have their respective energies bounded from below. Up to now such a construction was missing in the literature. Aim of this paper is to fill this gap.

The plan of this paper is as follows: In Section 2 we present some fundamentals of 't Hooft's pre-quantization method. Section 3 is dedicated to non-interacting emergent systems. In particular, we discuss a simple example of a be-able dynamics that gives rise, at an emergent level, to two non-interacting QM oscillators. A nontrivial extension to interacting oscillators is presented in Section 4. There we show how two Bateman oscillators can give rise to a composite QM system — an *isotonic* oscillator, which, in the limit of large separation between subparts, decouples into two non-interacting QM oscillators. Conclusions are in Section 5.

2. 't Hooft Pre-Quantization Method

In this section we outline 't Hooft's continuous-time pre-quantization method.[6] To this end we start with the assumption that the dynamics at the primordial deterministic level is described by the Hamiltonian

$$H = \sum_i p_i f_i(\mathbf{q}) + g(\mathbf{q}). \tag{1}$$

Here f_i and g are functions of $\mathbf{q} = \{q_1, \ldots, q_n\}$. Note that the equation of motion for q_i, i.e.

$$\dot{q}_i = f_i(\mathbf{q}), \tag{2}$$

is autonomous, since the p_i variables are decoupled. The above system is obviously deterministic, however its Hamiltonian is not bounded from below. Because of the autonomous character of the dynamical equation (2) one can define a formal Hilbert space $\mathcal{H}$ spanned by the states $\{|\mathbf{q}\rangle; \mathbf{q} \in \mathbb{R}^n\}$, and associate with p_i the operator $\hat{p}_i = -i\partial/\partial q_i$. It is not difficult to see that the generator of time translations, i.e. the

"Hamiltonian" operator $\hat{H} = \sum_i \hat{p}_i f_i(\hat{\mathbf{q}}) + g(\hat{\mathbf{q}})$ generates precisely the operatorial version of the *deterministic* evolution equation (2) (in Heisenberg picture),

$$\dot{\hat{q}}_i = f_i(\hat{\mathbf{q}})\,. \tag{3}$$

In other words, the operators $\hat{q}_i$ evolve deterministically even after "quantization". This evolution is only between base vectors. From the Schrödinger picture point of view this means that the *state vector* evolves smoothly from one *base vector* to another. Because of this, there is *no* (non-trivial) linear superposition of the state vector in terms of base vectors and hence *no* interference phenomenon shows up when measurements of **q**-variables – the be-ables – are performed.

The Hamiltonian (1) is unbounded from below. In principle, this fact could create problems with physics: partition functions, equilibrium states, etc. all require in general bounded Hamiltonians. However, this should not disturb us too much since the Hamiltonian (1) governs (via an operatorial formulation) the be-able dynamics, not the observables dynamics. On the contrary, the *emergent* Hamiltonian, describing the *physical observables*, must be bounded from below. The lower bound is, in the 't Hooft proposal, just an emergent property formed during the coarse graining of the be-able degrees of freedom down to the observational ones, and it will be constructed by implementing the information loss constraint (ILC).

We review now a simple mechanism showing how a lower bound for H may develop. Consider $\rho(\hat{q})$ to be some positive function of $\hat{q}_i$ (but not $\hat{\mathbf{p}}$) with $[\hat{\rho}, \hat{H}] = 0$. One then defines the splitting

$$\hat{H} \;=\; \hat{H}_+ - \hat{H}_-, \qquad \hat{H}_+ \;=\; \left(\hat{\rho} + \hat{H}\right)^2 \frac{\hat{\rho}^{-1}}{4}, \quad \hat{H}_- \;=\; \left(\hat{\rho} - \hat{H}\right)^2 \frac{\hat{\rho}^{-1}}{4}, \tag{4}$$

where $\hat{H}_+$ and $\hat{H}_-$ are positive-definite operators satisfying

$$[\hat{H}_+, \hat{H}_-] = [\hat{\rho}, \hat{H}_\pm] = 0\,. \tag{5}$$

At this point we introduce the "coarse-graining" operator $\hat{\Phi}$ that describes the loss of information that is happening during the passage from the be-able to observational scale. The operator originally considered by 't Hooft[1,6] is

$$\hat{\Phi} = \hat{H}_-\,. \tag{6}$$

The operator $\hat{\Phi}$ is then implemented as a constraint on the Hilbert space $\mathcal{H}$. The observed states, which we call *physical* states $|\psi\rangle_{phys}$ are given by the condition

$$\hat{\Phi}|\psi\rangle_{phys} = \hat{H}_-|\psi\rangle_{phys} = 0\,. \tag{7}$$

This equation identifies the states that are still distinguishable at the observational scale[a]. The resulting physical state space, i.e. the space of observables, has the

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[a]For a discussion about the Dirac classification of constraint (6) and on the gauge transformations generated by $\hat{\Phi}$ see Ref. 9

energy eigenvalues that are trivially bounded from below, thus in the Schrödinger picture the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle_{phys} = -i\hat{H}_+|\psi_t\rangle_{phys}\,, \tag{8}$$

has only positive frequencies on physical states.

3. Two Non-Interacting QM Oscillators

In this section we discuss a simple example of a be-able dynamics giving rise at an emergent level to two non interacting quantum oscillators. Our exposition closely follows the procedure presented in Ref. 7. We start with a primordial dynamics described by two (classical) Bateman oscillators (BO's). The system of two decoupled BO's is described by the equations of motion ($i = A, B$)

$$m_i\ddot{x}_i + \gamma_i\dot{x}_i + \kappa_i x_i = 0\,, \qquad m_i\ddot{y}_i - \gamma_i\dot{y}_i + \kappa_i y_i = 0\,, \tag{9}$$

where $m_i = (m_A, m_B)$, $\gamma_i = (\gamma_A, \gamma_B)$ and $\kappa_i = (\kappa_A, \kappa_B)$. This suggests that under appropriate boundary conditions the y_i–oscillator is the time–reversed image of the x_i–oscillator. In order to show that the Hamiltonians H_i for the systems (9) belongs to the class of 't Hooft's Hamiltonians, it is convenient to rewrite the former system in a rotated coordinate frame, i.e. $x_{1i} = (x_i + y_i)/\sqrt{2}$, $x_{2i} = (x_i - y_i)/\sqrt{2}$. Then the corresponding ith Hamiltonian reads

$$H_i = \frac{1}{2m_i}(p_{1i}^2 - p_{2i}^2) - \frac{\gamma_i}{2m_i}(p_{1i}x_{2i} + p_{2i}x_{1i}) + \frac{1}{2}\left(\kappa_i - \frac{\gamma_i^2}{4m_i}\right)(x_{1i}^2 - x_{2i}^2)\,. \tag{10}$$

The algebraic structure for the total system $H_T = H_A + H_B$ is the one of $su(1,1)\otimes su(1,1)$. Indeed, from the dynamical variables $p_{\alpha i}$ and $x_{\alpha i}$ one may construct the functions

$$J_{1i} = \frac{1}{2m_i\Omega_i}\,p_{1i}p_{2i} - \frac{m_i\Omega_i}{2}\,x_{1i}x_{2i}\,, \qquad J_{2i} = \frac{1}{2}(p_{1i}x_{2i} + p_{2i}x_{1i})\,,$$

$$J_{3i} = \frac{1}{4m_i\Omega_i}\left(p_{1i}^2 + p_{2i}^2\right) + \frac{m_i\Omega_i}{4}\left(x_{1i}^2 + x_{2i}^2\right)\,. \tag{11}$$

Here $\Omega_i = \sqrt{\frac{1}{m_i}(\kappa_i - \frac{\gamma_i^2}{4m_i})}$, with $\kappa_i > \frac{\gamma_i^2}{4m_i}$. The quadratic Casimirs for the Poisson algebra of the functions (11) are defined as $\mathcal{C}_i^2 = J_{3i}^2 - J_{2i}^2 - J_{1i}^2$. The $\mathcal{C}_i$ explicitly read as

$$\mathcal{C}_i = \frac{1}{4m_i\Omega_i}[(p_{1i}^2 - p_{2i}^2) + m_i^2\Omega_i^2(x_{1i}^2 - x_{2i}^2)]\,. \tag{12}$$

In terms of J_{2i} and $\mathcal{C}_i$ the Hamiltonians H_i can be formulated as

$$H_i = 2\left(\Omega_i\mathcal{C}_i - \Gamma_i J_{2i}\right)\,, \tag{13}$$

with $\Gamma_i = \gamma_i/2m_i$. Eq.(13) indicates that H_i are now in the 't Hooft form, with the $\mathcal{C}_i$ and J_{2i} playing the role of p's, and the Ω_i and Γ_i that of $f(\mathbf{q})$'s. By introducing

hyperbolic coordinates, i.e. $x_{1i} = r_i \cosh u_i$, $x_{2i} = r_i \sinh u_i$, $(r_i \in \mathbb{R} ; u_i \in \mathbb{R})$, one can cast J_{2i} and $\mathcal{C}_i$ in a particularly simple form (See Ref. 7).

To consider the *operatorial* description of this be-able dynamics, we promote all the relevant quantities, H_i, $\mathcal{C}_i$, J_{2i} and $\mathbf{q}_i$ to operators. We can then properly speak about their commutators, and we note that not only $\mathbf{q}_i$ are be-ables but also (and this is specific for Bateman's system) $\mathcal{C}_i$ and J_{2i} are be-ables (in fact they all commute among themselves, and, of course, with H_i). 't Hooft's ILC can now be enforced according to prescription (4), (6), and (7). Choosing $\hat{\rho} = 2\Omega_i\hat{\mathcal{C}}_i$, and taking $\hat{\mathcal{C}}_i > 0$ (this can be done, because $\hat{\mathcal{C}}_i$ are constants of motion), the splitting reads

$$\hat{H}_{i+} = \frac{1}{2\Omega_i\hat{\mathcal{C}}_i}(2\Omega_i\hat{\mathcal{C}}_i - \Gamma_i\hat{J}_{2i})^2, \qquad \hat{H}_{i-} = \frac{1}{2\Omega_i\hat{\mathcal{C}}_i}\Gamma_i^2\hat{J}_{2i}^2. \qquad (14)$$

The quantization emerges after one enforces ILC, i.e., $\hat{H}_{i-}|\psi\rangle_{phys} = 0$, $i = A, B$ which defines/selects the physical states. The same constraint is equivalent to $\hat{J}_{2i}|\psi\rangle_{phys} = 0$, and thus allows us to get the lower bound for the Hamiltonian by projecting out the states responsible for the negative part of the spectrum. In fact this implies $\hat{H}_i|\psi\rangle_{phys} = \hat{H}_{i+}|\psi\rangle_{phys} = 2\Omega_i\hat{\mathcal{C}}_i|\psi\rangle_{phys}$, and

$$2\Omega_i\hat{\mathcal{C}}_i|\psi\rangle_{phys} = \left[\frac{1}{2m_i}\big(\hat{p}_{r_i}^2 + m_i^2\Omega_i^2\hat{r}_i^2\big) - \frac{2\hat{J}_{2i}^2}{m_i\hat{r}_i^2}\right]|\psi\rangle_{phys} = \left(\frac{\hat{p}_{r_i}^2}{2m_i} + \frac{m_i}{2}\Omega_i^2\hat{r}_i^2\right)|\psi\rangle_{phys}.$$

We have thus reproduced two independent quantum mechanical oscillators on the physical states $|\psi\rangle_{phys}$. Each of the Hamiltonians H_A and H_B reduces independently to the Hamiltonian of a QM oscillator. Two independent BO's give rise to two independent QM oscillators.

4. Composite QM System

So far, we have only doubled the system already studied in Ref. 7. Now we wish to show how the implementation of 't Hooft's ILC on the global system dictates the form of the interaction between the emergent QM oscillators. To this end we consider together, as a single composite system, the two BO's (10). The total be-able Hamiltonian can be written as

$$H_T = H_A + H_B = 2(\Omega_A\mathcal{C}_A + \Omega_B\mathcal{C}_B) - 2(\Gamma_A J_{2A} + \Gamma_B J_{2B}). \qquad (15)$$

Since $\mathcal{C}_i$ are constants of motion (being Casimirs of the respective $su(1,1)$ algebras), this ensures that once they are chosen to be positive (as we do from now on), they remain such at all times. Also J_{2i} are constants of motion ($\{H_i, J_{2i}\} = 0$). We can therefore define new integrals of motion:

$$\mathcal{C} \equiv \frac{\Omega_A\mathcal{C}_A + \Omega_B\mathcal{C}_B}{\Omega}, \quad J \equiv \frac{\Gamma_A J_{2A} + \Gamma_B J_{2B}}{\Gamma}, \qquad (16)$$

where Ω and Γ are opportunely defined numbers. Using the fact that $\Omega_i > 0$ and assuming that $\Omega > 0$, we may conclude that $\mathcal{C}_A$, $\mathcal{C}_B > 0 \Rightarrow \mathcal{C} > 0$. The positivity

of $\mathcal{C}$ is guaranteed by our choice $\mathcal{C}_i > 0$. The total Hamiltonian will be

$$H_T = 2\Omega\mathcal{C} - 2\Gamma J. \tag{17}$$

With the choice $\rho = 2\Omega\mathcal{C}$, H_T can be split *à la* (14) as

$$H_+ = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J)^2, \qquad H_- = \frac{1}{2\Omega\mathcal{C}}\Gamma^2 J^2. \tag{18}$$

Given that $\rho = 2\Omega\mathcal{C}$, then ρ is a positive integral of motion. We note that $\mathcal{C}$, J are again be-ables because they are functions of be-ables. We impose the constraint on the observational scale in the form

$$\hat{H}_-|\psi\rangle_{phys} = \hat{J}|\psi\rangle_{phys} = 0. \tag{19}$$

This implies $\hat{H}_T \approx \hat{H}_+ \approx 2\Omega\hat{\mathcal{C}}$, ($\approx$ indicates that operators are equal only on the physical states). Since $\hat{J} = (\Gamma_A\hat{J}_{2A} + \Gamma_B\hat{J}_{2B})/\Gamma$, the condition $\hat{J} \approx 0$ implies a relation between $\hat{J}_{2A}$ and $\hat{J}_{2B}$ which in turn implies an correlation between $\hat{H}_A$ and $\hat{H}_B$. Solving with respect to $\hat{J}_{2B}$, Eq.(19) gives $\hat{J}_{2B} \approx -\Gamma_A\hat{J}_{2A}/\Gamma_B$, which, when substituted into $\hat{H}_T = 2\Omega\hat{\mathcal{C}} - 2\Gamma\hat{J}$, yields (using hyperbolic coordinates)

$$\hat{H}_T \approx \hat{H}_+ \tag{20}$$
$$\approx \left(\frac{\hat{p}_{r_A}^2}{2m_A} - \frac{2\hat{J}_{2A}^2}{m_A\,\hat{r}_A^2} + \frac{1}{2}m_A\,\Omega_A^2\hat{r}_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2}m_B\,\Omega_B^2\hat{r}_B^2\right) - \frac{2}{m_B}\frac{\Gamma_A^2}{\Gamma_B^2}\frac{1}{\hat{r}_B^2}\,\hat{J}_{2A}^2.$$

The emergent Hamiltonian H_T, Eq.(20), represents a good quantum system, since, by its very construction, it is bounded from below on the physical states. The term inside the first parenthesis is simply $2\,\Omega_A\hat{\mathcal{C}}_A$. Such a term is constant, because $\hat{\mathcal{C}}_A$ is an integral of motion. The second term represents a QM oscillator, while the third corresponds to a centripetal barrier. The inverse square potential $1/r^2$ is analogous to the centrifugal contribution in polar coordinates and one may thus expect an exact solvability. The only difference here is that $r \in \mathbb{R}$ and not merely $\mathbb{R}^+$.

The system with the Hamiltonian

$$H = \frac{N^2}{2}\,p_{r_B}^2 + \frac{Q^2}{2}\,r_B^2 + \frac{R^2 - N^2/4}{2r_B^2}, \tag{21}$$

and $r_B \in \mathbb{R}$, $N, Q, R \in \mathbb{R}^+$, is known as the *isotonic* oscillator.[13] Its spectrum can be exactly solved by purely algebraic means, since the Hamiltonian admits a shape-invariant factorization.[14] In our case the eigenvalue equation for $\hat{J}_{2A}$ reads $\hat{J}_{2A}|\Psi_{j,\mu}\rangle = \mu_A|\Psi_{j,\mu}\rangle$, and since we have

$$N^2 = \frac{1}{m_B}, \quad Q^2 = m_B\Omega_B^2, \quad R^2/N^2 = \frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B}\mu_A\right)^2,$$

the actual spectrum of (20) is

$$E_{n,\mp}(c,\mu_A) = \Omega_B\left(2n\,\mp\,\sqrt{\frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B}\mu_A\right)^2} + 1\right) + c, \qquad n \in \mathbb{N}, \tag{22}$$

where c is a shift constant term due to the presence of $2\Omega_A\hat{\mathcal{C}}_A$. Note that when $2\Gamma_A\mu_A/\Gamma_B \ll 1/2$, the inverse square potential in (20) can be neglected and the system reduces to that of a QM linear oscillator with a shift term c. This follows also directly from the spectrum (22) provided we set $\Gamma_A = 0$ and consider both signs.

A further important interpretation of the emergent system (20) can be obtained when μ_A is an imaginary number, i.e., when $\hat{J}_{2A}$ belongs to a non-unitary realization of the D_j series. Then $R/N > 1/2$, the potential in (21) is repulsive at the origin, and the motion takes place only in the domain $r_B \geq 0$. This, in particular, allows to view r_B as a radial coordinate and the inverse square potential in (20), (21) as a rotationally invariant interaction of the spin-orbit type. To see this we rewrite the interaction potential in (20) in the form

$$\hat{H}_{int} \approx \frac{2}{m_B}\frac{\Gamma_A}{\Gamma_B}\frac{1}{\hat{r}_B}\frac{\partial V}{\partial r_B}(\hat{\mathbf{J}}_B \cdot \hat{\mathbf{J}}_A), \qquad V = \log r_B. \tag{23}$$

Here we have used $\hat{J}_{2B} \approx -\Gamma_A\hat{J}_{2A}/\Gamma_B$. We now formally identify $\mathbf{J}_B = i\mathbf{L}$ and $\mathbf{J}_A = i\mathbf{S}$, where $\mathbf{L}$ plays the role of the orbital angular momentum of the "particle" B and $\mathbf{S}$ plays the role of its "spin". With this, the interaction energy reads

$$H_{int} = -\frac{2}{m_B}\frac{\Gamma_A}{\Gamma_B}\frac{1}{r_B}\frac{\partial V}{\partial r_B}(\mathbf{L} \cdot \mathbf{S}) \equiv -\frac{g}{2m_Bc^2}\frac{1}{r_B}\frac{\partial V}{\partial r_B}(\mathbf{L} \cdot \mathbf{S}), \tag{24}$$

where we have identified Γ_A/Γ_B with $g/4c^2$, i.e. with a quarter of the gyromagnetic factor. The function V thus plays the role of the planar radial scalar electromagnetic potential. So, the emergent QM system can be alternatively interpreted also as a particle in a magnetic field, with a spin-orbital interaction term.

Far from the interaction region (large r_A, r_B), the interaction is switched off and the asymptotic emergent QM Hamiltonian is

$$\hat{H}_T \approx \hat{H}_+ \approx \left(\frac{\hat{p}_{r_A}^2}{2m_A} + \frac{1}{2}m_A\Omega_A^2\hat{r}_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2}m_B\Omega_B^2\hat{r}_B^2\right) \tag{25}$$

The total system and its subsystems are still genuine QM systems because both the oscillators are bounded from below. Therefore, far from the interaction region we have two non-interacting QM oscillators. Let us finally mention two points. Firstly: before imposing the ILC on the global system, we have two non interacting, independent quantum oscillators. Only the enforcement of the dissipation constraint on the global system gives rise to the interaction $J_{2A} \leftrightarrow J_{2B}$. So the constraint dictates the interaction. Secondly, the interaction term depends on the dissipations constants Γ_A, Γ_B. In particular the interaction can be switched off by setting $\Gamma_A \to 0$. So the dissipation plays a key rôle in the interaction.

5. Conclusions

Our *modus operandi* here was based on 't Hooft's quantization proposal. With this we have tried to convey two main messages: Firstly, that quantum mechanics does

not really preclude the existence of an underlying explanatory basis; and secondly, that it is not difficult to formulate a genuine quantum mechanical composite system which originates from an underlying deterministic dynamics, and yet where both the total system and its identifiable sub-parts have their respective energies bounded from below. Our points were illustrated with a be-able dynamics described by two Bateman oscillators. On an emergent level they give rise to a composite QM system — an isotonic oscillator, which in the limit of large separation between subparts, decouples into two non-interacting QM oscillators.

Acknowledgments

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