Dissipation and quantization for composite systems

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A R T I C L E   I N F O

A B S T R A C T

In the framework of 't Hooft's quantization proposal, we show how to obtain from the composite system of two classical Bateman's oscillators a quantum isotonic oscillator. In a specific range of parameters, such a system can be interpreted as a particle in an effective magnetic field, interacting through a spin–orbit interaction term. In the limit of a large separation from the interaction region one can describe the system in terms of two irreducible elementary subsystems which correspond to two independent quantum harmonic oscillators.

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1. Introduction

In a series of papers [1,2] Gerard 't Hooft has put forward the conjecture that the origin of the quantum mechanical nature of our world is in the dissipation of information which should occur at very high energies (Planck scale) in a regime of deterministic dynamics.

The basic observation of 't Hooft is that there exists a family of dynamical systems that can be described by means of Hilbert space techniques without loosing their deterministic character. Only after enforcing certain constraints expressing information loss, one obtains *bona fide* quantum systems. In this picture the quantum states of actually observed degrees of freedom (observables) can be identified with equivalence classes of states that span the original (primordial) Hilbert space of truly existing degrees of freedom (be-ables). It is important to remark that be-ables are not referring to conventional macroscopic variables, such as a pointer on a detection device, but rather to a set of what 't Hooft calls "primordial" variables. Conventional variables, like mass, energy, position, etc., are viewed as emergent (non-primordial) degrees of freedom that are described in terms of states in a quotient Hilbert space.

't Hooft's proposal has been discussed in a number of papers [3–7] and explicit examples of the above scheme have been constructed, e.g., in Refs. [2,8–13]. In particular, in Ref. [8] the Hamiltonian for a couple of classical damped-amplified oscillators [14], also known as Bateman's (dual) oscillator, has been shown to be a suitable deterministic system giving rise to a genuine quantum harmonic oscillator, once the information loss condition had been implemented.

In the explicit realizations of 't Hooft's quantization scheme so far considered, only non-interacting quantum systems have been constructed. It is thus important to study deterministic systems which would give rise to interacting quantum systems, after appropriate constraints are imposed. In this Letter, we accomplish this task and consider the composite (deterministic) system made of two couples of classical damped-amplified oscillators. Of course, the results of Ref. [8] apply to each of Bateman's oscillators separately. We show that the condition of information loss (ILC) when applied to the global system composed of two Bateman's oscillators determines the form of the interaction of the resulting quantum system, which turns out to be a quantum isotonic oscillator [15]. In a specific range of parameters, the system can be also interpreted as a particle in an effective magnetic field interacting through a spin–orbit interaction term.

The plan of the Letter is the following. As preliminaries, in Sections 2 and 3 we summarize the main features of 't Hooft's proposal and of the formalism introduced in Ref. [8], respectively. This will turn out to be useful for the subsequent discussion in Section 4, where we show how the constraint of information loss

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imposed on the global system dictates the interaction between the component subsystems. Section 5 is devoted to final remarks and conclusions. Some further mathematical details are confined into the appendices.

2. 't Hooft's quantization scheme

In this section we briefly outline 't Hooft's continuous-time quantization proposal [2]. To this end we consider the dynamics at the primordial, deterministic, level as described by the equations

$$\dot{q}_i = f_i(\mathbf{q}), \tag{1}$$

with a vector field f_i on some configuration space $Q \subseteq \mathbb{R}^n$. The system (1) is generally non-Hamiltonian but it can be lifted [16] to a Hamiltonian system on the phase space $\Gamma = Q \times \mathbb{R}^n$, provided one defines the Hamiltonian $H : \Gamma \mapsto \mathbb{R}$ as

$$H = \sum_i p_i f_i(\mathbf{q}) + g(\mathbf{q}). \tag{2}$$

Here $(\mathbf{p}, \mathbf{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ denote the canonical coordinates on Γ and g is certain function on Q to be discussed shortly. Note that, due to the particular (linear) form of the Hamiltonian in the p_i variables, the system described by Eq. (1) is autonomous in the q_i variables.

We have:

$$q_i(t + \Delta t) = q_i(t) + f_i(\mathbf{q}) \Delta t + \frac{1}{2} f_k(\mathbf{q}) \frac{\partial^2 H}{\partial p_i \partial q_k} (\Delta t)^2 + \cdots$$
$$= F_i(\mathbf{q}(t), \Delta t), \tag{3}$$

where F_i is some function of $\mathbf{q}(t)$ and Δt but not $\mathbf{p}$. Since (3) holds for any Δt we get the Poisson bracket

$$\{q_i(t'), q_k(t)\} = 0, \tag{4}$$

which holds for any t and t'.

Because of the autonomous character of the dynamical equation (1) one can define a formal Hilbert space $\mathcal{H}$ spanned by the states $\{|\mathbf{q}\rangle; \mathbf{q} \in \mathbb{R}^n\}$, and associate with p_i the operator $\hat{p}_i = -i\partial/\partial q_i$. It is not difficult to see that the generator of time translations, the Hamiltonian operator, of the form $\hat{H} = \sum_i \hat{p}_i f_i(\hat{\mathbf{q}}) + g(\hat{\mathbf{q}})$ generates the *deterministic* evolution equation (1) (see note [17]). Indeed, we firstly observe that because $\hat{H}$ is generator of time translations then in the Heisenberg picture

$$\hat{q}_i(t + \Delta t) = e^{i\Delta t \hat{H}} \hat{q}_i(t) e^{-i\Delta t \hat{H}}, \tag{5}$$

which for infinitesimal Δt implies

$$\hat{q}_i(t + \Delta t) - \hat{q}_i(t) = i\Delta t [\hat{H}, q_i(t)] \quad \Rightarrow \quad \dot{\hat{q}}_i = f_i(\hat{\mathbf{q}}). \tag{6}$$

On the other hand, for arbitrary finite Δt we have from (5)

$$\hat{q}_i(t + \Delta t) = \sum_{m=0}^{\infty} \frac{1}{m!} [\hat{H}, [\hat{H}, [\cdots [\hat{H}, \hat{q}_i(t)]] \cdots]]$$
$$= \tilde{F}_i(\hat{\mathbf{q}}(t), \Delta t). \tag{7}$$

On the first line $\hat{H}$ appears in the generic term of the sum m times. On the second line $\tilde{F}_i$ is some function of $\hat{\mathbf{q}}(t)$ and Δt but not $\hat{\mathbf{p}}$ which immediately implies that

$$[\hat{q}_i(t), \hat{q}_j(t')] = 0, \tag{8}$$

for any t and t' (this in turn gives $F_i = \tilde{F}_i$). Result (8) shows that the Heisenberg equation of the motion for $\hat{q}_i(t)$ in the $\mathbf{q}$-representation is identical with the c-number dynamical equation (1). This is because $\hat{q}_i(t + \Delta t)$ and $\hat{q}_i(t)$ commute, and hence $\hat{q}_i(t + \Delta t)$, $\hat{q}_i(t)$, $f_i(\hat{\mathbf{q}})$ and also $d\hat{q}_i(t)/dt$ can be simultaneously

diagonalized. In this diagonal basis we get back the c-number autonomous equation (1). From the Schrödinger picture point of view this means that the *state vector* evolves smoothly from one *base vector* to another (in the Schrödinger picture base vectors are time independent and fixed). So at each instant the state vector coincides with some specific base vector. Because of this, there is *no* non-trivial linear superposition of the state vector in terms of base vectors and hence *no* interference phenomenon shows up when measurement of $\mathbf{q}$-variable is performed. In other words, the operators $\hat{q}_i$ evolve deterministically even after "quantization".

Dynamical variables fulfilling Eq. (8) were first considered by Bell [18] who called them *be-ables* as opposed to observed dynamical variables which in QM are called *observables*.

The Hamiltonian (2) is unbounded from below. This fact is not really a problem since the dynamics of be-ables is actually described by Eq. (1). However, at the observational (emergent) level Hamiltonians are key objects as they are tightly connected with the concept of energy of the system and so they must be bounded from below.

In order to introduce a lower bound for the Hamiltonian $\hat{H}$, we consider a positive function of the $\hat{q}_i$ alone, $\rho(\hat{q})$, commuting with $\hat{H}$: $[\hat{\rho}, \hat{H}] = 0$. We can then write

$$\hat{H} = \hat{H}_+ - \hat{H}_-,$$
$$\hat{H}_+ = \frac{1}{4\hat{\rho}} (\hat{\rho} + \hat{H})^2, \qquad \hat{H}_- = \frac{1}{4\hat{\rho}} (\hat{\rho} - \hat{H})^2, \tag{9}$$

where $\hat{H}_+$ and $\hat{H}_-$ are positive-definite operators satisfying

$$[\hat{H}_+, \hat{H}_-] = [\hat{\rho}, \hat{H}_\pm] = 0. \tag{10}$$

One then requires that the states at the observational level, also called *physical* states $|\psi\rangle_{\text{phys}}$, must satisfy the constraint [1,2]

$$\hat{H}_- |\psi\rangle_{\text{phys}} = 0, \tag{11}$$

which will be henceforth referred to as the *information loss condition* (ILC).

It is interesting to interpret, in full generality, such a constraint as a "coarse-graining" operation induced by an operator $\hat{\Phi}$ describing the process of information loss occurring when passing from the be-able to the observational level (see Section 5 for further remarks on $\hat{\Phi}$). In the case of Eq. (11), we have

$$\hat{\Phi} = \hat{H}_-. \tag{12}$$

Such a constraint is, according to Dirac's classification [19], a first class primary constraint because $[\hat{\Phi}, \hat{\Phi}] = 0$ and $[\hat{\Phi}, \hat{H}] = 0$. First-class constraints generate gauge transformation and produce equivalence classes of states [19], which are generally non-local. States belong to the same class, even if space-like separated, when they can be transformed into each other by gauge transformations generated by $\hat{\Phi}$. Let $\mathcal{G}$ be the group of these gauge transformations.

The above equivalence classes represent the physical states, namely the ones accessible at observational level in Quantum Mechanics. Denoting with $\mathcal{O}$ the space spanned by the observables, we identify it with the quotient space

$$\mathcal{O} = \mathcal{H}_c / \mathcal{G}. \tag{13}$$

From Eq. (11), we see that physical states have positive energy spectrum, since

$$\hat{H} |\psi\rangle_{\text{phys}} = \hat{H}_+ |\psi\rangle_{\text{phys}} = \hat{\rho} |\psi\rangle_{\text{phys}}. \tag{14}$$

Thus, in the Schrödinger picture the equation of motion

$$\frac{d}{dt} |\psi_t\rangle_{\text{phys}} = -i\hat{H}_+ |\psi_t\rangle_{\text{phys}} \tag{15}$$

has only positive frequencies on physical states.

In Ref. [2] 't Hooft observed that when the dynamical equations (1) describe configuration-space chaotic dynamical systems, the equivalence classes could be related to their stable orbits (e.g., limit cycles). Information loss is responsible for clustering of trajectories to equivalence classes: after a while one cannot retrace back the initial conditions of a given trajectory; one can only say to which attractor the evolution leads. Applications of the outlined canonical scenario were given, e.g., in Refs. [8–10].

3. Two Bateman's dual oscillators

A system of damped-amplified harmonic oscillators (Bateman's dual oscillator) was considered in Ref. [8] as an explicit example yielding 't Hooft's be-able dynamics. There, the ILC − Eq. (11), gave rise to a genuine one-dimensional quantum harmonic oscillator.

In the present section we consider two Bateman's oscillators, labeled by the index $i = A, B$:

$$m_i \ddot{x}_i + \gamma_i \dot{x}_i + \kappa_i x_i = 0, \tag{16}$$

$$m_i \ddot{y}_i - \gamma_i \dot{y}_i + \kappa_i y_i = 0, \tag{17}$$

where $m_i = (m_A, m_B)$, $\gamma_i = (\gamma_A, \gamma_B)$ and $\kappa_i = (\kappa_A, \kappa_B)$. The y_i-oscillator is the time-reversed image of the x_i-oscillator. In the following, as a preliminary to the discussion of Section 4, we will closely follow the treatment presented in Ref. [8], where the reader can find details here omitted for brevity.

The Lagrangian for the ith-couple of oscillators is

$$\mathcal{L}_i = m_i \dot{x}_i \dot{y}_i + \frac{\gamma_i}{2} (x_i \dot{y}_i - \dot{x}_i y_i) - \kappa_i x_i y_i, \tag{18}$$

and the conjugated momenta are

$$p_{x_i} = \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i} = m_i \dot{y}_i - \frac{1}{2} \gamma_i y_i,$$

$$p_{y_i} = \frac{\partial \mathcal{L}_i}{\partial \dot{y}_i} = m_i \dot{x}_i + \frac{1}{2} \gamma_i x_i. \tag{19}$$

Therefore the Hamiltonian for ith oscillator reads

$$H_i = \frac{1}{m_i} p_{x_i} p_{y_i} + \frac{\gamma_i}{2m_i} (y_i p_{y_i} - x_i p_{x_i}) + \left(\kappa_i - \frac{\gamma_i^2}{4m_i} \right) x_i y_i. \tag{20}$$

In order to show that H_i belongs to the class of 't Hooft's Hamiltonians, it is convenient to reformulate the former system in a rotated coordinate frame, i.e.

$$x_{1i} = \frac{x_i + y_i}{\sqrt{2}}, \qquad x_{2i} = \frac{x_i - y_i}{\sqrt{2}}. \tag{21}$$

In these new coordinates the ith Lagrangian has the form

$$\mathcal{L}_i = \frac{m_i}{2} (\dot{x}_{1i}^2 - \dot{x}_{2i}^2) + \frac{\gamma_i}{2} (\dot{x}_{1i} x_{2i} - x_{1i} \dot{x}_{2i}) - \frac{\kappa_i}{2} (x_{1i}^2 - x_{2i}^2). \tag{22}$$

For the new canonical momenta $p_{\alpha i} = \partial \mathcal{L} / \partial \dot{x}_{\alpha i}$, where $x_{\alpha i} \equiv (x_{1i}, x_{2i})$ and $p_{\alpha i} \equiv (p_{1i}, p_{2i})$, we obtain

$$p_{1i} = m_i \dot{x}_{1i} + \frac{1}{2} \gamma_i x_{2i},$$

$$p_{2i} = -m_i \dot{x}_{2i} - \frac{1}{2} \gamma_i x_{1i}, \tag{23}$$

and thus the corresponding ith Hamiltonian reads

$$H_i = \frac{1}{2m_i} (p_{1i}^2 - p_{2i}^2) - \frac{\gamma_i}{2m_i} (p_{1i} x_{2i} + p_{2i} x_{1i})$$
$$+ \frac{1}{2} \left(\kappa_i - \frac{\gamma_i^2}{4m_i} \right) (x_{1i}^2 - x_{2i}^2). \tag{24}$$

The algebraic structure for the total system $H_T = H_A + H_B$ is the one of $su(1,1) \otimes su(1,1)$. Indeed, from the dynamical variables $p_{\alpha i}$ and $x_{\alpha i}$ one may construct the functions

$$J_{1i} = \frac{1}{2m_i \Omega_i} p_{1i} p_{2i} - \frac{m_i \Omega_i}{2} x_{1i} x_{2i},$$

$$J_{2i} = \frac{1}{2} (p_{1i} x_{2i} + p_{2i} x_{1i}),$$

$$J_{3i} = \frac{1}{4m_i \Omega_i} (p_{1i}^2 + p_{2i}^2) + \frac{m_i \Omega_i}{4} (x_{1i}^2 + x_{2i}^2), \tag{25}$$

where $\Omega_i = \sqrt{\frac{1}{m_i} (\kappa_i - \frac{\gamma_i^2}{4m_i})}$, and $\kappa_i > \frac{\gamma_i^2}{4m_i}$. Applying now the canonical Poisson brackets $\{x_{\alpha i}, p_{\beta j}\} = \delta_{\alpha \beta} \delta_{ij}$ we obtain Poisson's subalgebra

$$\{J_{2i}, J_{3i}\} = J_{1i}, \qquad \{J_{3i}, J_{1i}\} = J_{2i},$$

$$\{J_{1i}, J_{2i}\} = -J_{3i}, \qquad \{J_{\alpha i}, J_{\beta j}\}|_{i \neq j} = 0. \tag{26}$$

The algebraic structure (26) corresponds to $su(1,1) \otimes su(1,1)$ algebra. The quadratic Casimirs for the algebra (26) are defined as

$$\mathcal{C}_i^2 = J_{3i}^2 - J_{2i}^2 - J_{1i}^2. \tag{27}$$

The $\mathcal{C}_i$ explicitly read

$$\mathcal{C}_i = \frac{1}{4m_i \Omega_i} \left[(p_{1i}^2 - p_{2i}^2) + m_i^2 \Omega_i^2 (x_{1i}^2 - x_{2i}^2) \right]. \tag{28}$$

In terms of J_{2i} and $\mathcal{C}_i$ the Hamiltonians H_i are given by

$$H_i = 2(\Omega_i \mathcal{C}_i - \Gamma_i J_{2i}), \tag{29}$$

where $\Gamma_i = \gamma_i / 2m_i$. Eq. (29) shows that H_i are of the 't Hooft form, with the $\mathcal{C}_i$ and J_{2i} playing the role of p's, and the Ω_i and Γ_i the one of $f(\mathbf{q})$'s.

A further simplification can be achieved by introducing the hyperbolic coordinates:

$$x_{1i} = r_i \cosh u_i,$$

$$x_{2i} = r_i \sinh u_i, \quad r_i \in \mathbb{R}; \ u_i \in \mathbb{R}. \tag{30}$$

The corresponding conjugated momenta then are [20,21]

$$p_{r_i} = p_{1i} \cosh u_i + p_{2i} \sinh u_i,$$

$$p_{u_i} = p_{1i} r_i \sinh u_i + p_{2i} r_i \cosh u_i. \tag{31}$$

In these coordinates J_{2i} and $\mathcal{C}_i$ have a particularly simple structure [8], namely

$$\mathcal{C}_i = \frac{1}{4\Omega_i m_i} \left[p_{r_i}^2 - \frac{1}{r_i^2} p_{u_i}^2 + m_i^2 \Omega_i^2 r_i^2 \right],$$

$$J_{2i} = \frac{1}{2} p_{u_i}. \tag{32}$$

In Ref. [8] it is shown that the following canonical transformations hold:

$$q_{1i} = \int \frac{dz_i m_i \Omega_i}{(4J_{2i}^2 + 4m_i \Omega_i \mathcal{C}_i z_i - m_i^2 \Omega_i^2 z_i^2)^{1/2}},$$

$$q_{2i} = 2u_i + \int \frac{dz_i}{z_i} \frac{2J_{2i}}{(4J_{2i}^2 + 4m_i \Omega_i \mathcal{C}_i z_i - m_i^2 \Omega_i^2 z_i^2)^{1/2}},$$

$$p_{1i} = \mathcal{C}_i, \qquad p_{2i} = J_{2i}, \tag{33}$$

with $z_i = r_i^2$. One has $\{\mathbf{q}_i, \mathbf{p}_i\} = 1$, and the other Poisson brackets vanishing.

Let us now consider the *operatorial* description of this be-able dynamics. To this end we promote all the relevant quantities, H_i, $\mathcal{C}_i$, J_{2i} and $\mathbf{q}_i$ to operators. Note that $(\mathbf{q}_i)$ and separately $(\mathcal{C}_i, J_{2i})$

are two independent sets of be-ables. The Casimir operators are [14,22]:

$$\hat{C}_i^2 = \hat{J}_{3i}^2 - \hat{J}_{2i}^2 - \hat{J}_{1i}^2 + \frac{1}{4}, \tag{34}$$

where the factor $1/4$ was introduce for convenience. Following 't Hooft, we now write the above Hamiltonians in the form

$$\hat{H}_i = \hat{H}_{i+} - \hat{H}_{i-},$$
$$\hat{H}_{i+} = \frac{1}{4\hat{\rho}_i}(\hat{\rho}_i + \hat{H}_i)^2, \qquad \hat{H}_{i-} = \frac{1}{4\hat{\rho}_i}(\hat{\rho}_i - \hat{H}_i)^2. \tag{35}$$

Choosing $\hat{\rho}_i = 2\Omega_i\hat{C}_i$, and taking $\hat{C}_i > 0$ (this can be done, because $\hat{C}_i$ are constants of motion), the splitting reads

$$\hat{H}_{i+} = \frac{(\hat{H}_i + 2\Omega_i\hat{C}_i)^2}{8\Omega_i\hat{C}_i} = \frac{1}{2\Omega_i\hat{C}_i}(2\Omega_i\hat{C}_i - \Gamma_i\hat{J}_{2i})^2,$$
$$\hat{H}_{i-} = \frac{(\hat{H}_i - 2\Omega_i\hat{C}_i)^2}{8\Omega_i\hat{C}_i} = \frac{1}{2\Omega_i\hat{C}_i}\Gamma_i^2\hat{J}_{2i}^2. \tag{36}$$

Quantization emerges after the information loss condition is imposed locally, i.e. separately on each of the Bateman oscillators:

$$\hat{J}_{2i}|\psi\rangle_{\mathrm{phys}} = 0, \tag{37}$$

which defines/selects the physical states and is equivalent to

$$\hat{H}_{i-}|\psi\rangle_{\mathrm{phys}} = 0, \quad i = A, B. \tag{38}$$

This implies

$$\hat{H}_i|\psi\rangle_{\mathrm{phys}} = (\hat{H}_{i+} - \hat{H}_{i-})|\psi\rangle_{\mathrm{phys}}$$
$$= \hat{H}_{i+}|\psi\rangle_{\mathrm{phys}} = 2\Omega_i\hat{C}_i|\psi\rangle_{\mathrm{phys}}, \tag{39}$$

and

$$2\Omega_i\hat{C}_i|\psi\rangle_{\mathrm{phys}} = \left[\frac{1}{2m_i}\left(\hat{p}_{r_i}^2 + m_i^2\Omega_i^2\hat{r}_i^2\right) - \frac{2\hat{J}_{2i}^2}{m_i\hat{r}_i^2}\right]|\psi\rangle_{\mathrm{phys}}$$
$$= \left(\frac{\hat{p}_{r_i}^2}{2m_i} + \frac{m_i}{2}\Omega_i^2\hat{r}_i^2\right)|\psi\rangle_{\mathrm{phys}}. \tag{40}$$

Eq. (40) reproduces, for each one of the systems A and B separately, the result of Ref. [8], namely each one of the Hamiltonians H_A and H_B reduces independently to the Hamiltonian of a QM oscillator.

In Appendix A we discuss some properties of the physical states $|\psi\rangle_{\mathrm{phys}}$.

4. The composite system

We now impose the ILC (11) to the composite system described by the Hamiltonian $H_T = H_A + H_B$, instead of imposing it to the systems H_A and H_B separately, as done in the previous section. We will find that the implementation of the information loss condition for the composite system dictates the form of the interaction between the component subsystems.

We start by writing the total Hamiltonian as

$$H_T = H_A + H_B$$
$$= 2(\Omega_A\mathcal{C}_A + \Omega_B\mathcal{C}_B) - 2(\Gamma_A J_{2A} + \Gamma_B J_{2B}). \tag{41}$$

The fact that the Casimirs $\mathcal{C}_i$ are constants of motion, guaranties that, once they are chosen to be positive (as we do from now on), they remain such at all times. Also J_{2i} are constants of motion ($\{H_i, J_{2i}\} = 0$). We can therefore define new integrals of motion:

$$\mathcal{C} \equiv \frac{\Omega_A\mathcal{C}_A + \Omega_B\mathcal{C}_B}{\Omega}, \qquad J \equiv \frac{\Gamma_A J_{2A} + \Gamma_B J_{2B}}{\Gamma}, \tag{42}$$

where Ω and Γ are quantities to be defined shortly. Using the fact that $\Omega_i > 0$ and assuming that $\Omega > 0$, we conclude that

$$\mathcal{C}_A, \mathcal{C}_B > 0 \quad \Rightarrow \quad \mathcal{C} > 0. \tag{43}$$

The positivity of $\mathcal{C}$ is guaranteed by our choice $\mathcal{C}_i > 0$. The Hamiltonian for the total system is

$$H_T = 2\Omega\mathcal{C} - 2\Gamma J, \tag{44}$$

and it reproduces exactly the Hamiltonian of each one of the two subsystems (cf. Eq. (29)). With the choice $\rho = 2\Omega\mathcal{C}$, H_T can be split as (cf. Eq. (36))

$$H_+ = \frac{(H_T + 2\Omega\mathcal{C})^2}{8\Omega\mathcal{C}} = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J)^2, \tag{45}$$

$$H_- = \frac{(H_T - 2\Omega\mathcal{C})^2}{8\Omega\mathcal{C}} = \frac{1}{2\Omega\mathcal{C}}\Gamma^2 J^2. \tag{46}$$

Note that $\rho \equiv 2\Omega\mathcal{C}$ is a positive integral of motion. $\mathcal{C}$ and J are again be-ables because they are functions of be-ables.

Now we switch to the operatorial description. Following 't Hooft, we impose the ILC on the observational scale in the form

$$\hat{H}_-|\psi\rangle_{\mathrm{phys}} = \hat{J}|\psi\rangle_{\mathrm{phys}} = 0. \tag{47}$$

This implies

$$\hat{H}_T \approx \hat{H}_+ \approx 2\Omega\hat{C}, \tag{48}$$

where $\approx$ indicates that operators are equal only on the physical states. Since $\hat{J} = (\Gamma_A\hat{J}_{2A} + \Gamma_B\hat{J}_{2B})/\Gamma$, the condition $\hat{J} \approx 0$ indicates that there must exist a relation between $\hat{J}_{2A}$ and $\hat{J}_{2B}$, which in turn implies a relation between $\hat{H}_A$ and $\hat{H}_B$. In other words, the global ILC establishes an interaction between the two Bateman's oscillators. We shall now study what kind of interaction this will induce. Solving with respect to $\hat{J}_{2B}$, Eq. (47) gives

$$\hat{J}_{2B} \approx -\frac{\Gamma_A}{\Gamma_B}\hat{J}_{2A}, \tag{49}$$

which, when substituted into H_T yields (see Eq. (32))

$$\hat{H}_T \approx \hat{H}_+$$
$$\approx \left(\frac{\hat{p}_{r_A}^2}{2m_A} - \frac{2\hat{J}_{2A}^2}{m_A\hat{r}_A^2} + \frac{1}{2}m_A\Omega_A^2\hat{r}_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2}m_B\Omega_B^2\hat{r}_B^2\right)$$
$$- \frac{2}{m_B}\frac{\Gamma_A^2}{\Gamma_B^2}\frac{1}{\hat{r}_B^2}\hat{J}_{2A}^2. \tag{50}$$

Note that the emergent Hamiltonian H_T in Eq. (50) is, by construction, bounded from below. The term inside the first parenthesis is $2\Omega_A\hat{C}_A$. Such a term is constant, because $\hat{C}_A$ is an integral of motion. The second term represents a QM oscillator, while the third corresponds to a centripetal barrier. The inverse square potential $1/r^2$ is analogous to the centrifugal contribution in polar coordinates and one may thus expect an exact solvability. The only difference here is that $r \in \mathbb{R}$ and not merely $\mathbb{R}^+$. The system with the Hamiltonian

$$H = \frac{N^2}{2}p_{r_B}^2 + \frac{Q^2}{2}r_B^2 + \frac{R^2 - N^2/4}{2r_B^2},$$
$$r_B \in \mathbb{R}, \ N, Q, R \in \mathbb{R}^+, \tag{51}$$

is known in quantum optics and in theory of coherent states [23,24] as the *isotonic* oscillator [15]. Its spectrum can be exactly solved by purely algebraic means since the Hamiltonian admits a shape-invariant factorization [25]. The energy eigenvalues read [25,26]

$$E_{n,\mp} = Q N\left(2n \mp \frac{R}{N} + 1\right), \quad n \in \mathbb{N}. \tag{52}$$

If $R/N \leqslant 1/2$, the potential is attractive in the origin, and both the negative and positive sign must be taken into account [26]. The positive sign in front of R/N has to be taken when $R/N > 1/2$. The inverse square potential is then repulsive at the origin, so the motion takes place only in the domain $r_B > 0$.

Since in our case

$$N^2 = \frac{1}{m_B}, \qquad Q^2 = m_B \Omega_B^2, \qquad \frac{R^2}{N^2} = \frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B}\mu_A\right)^2,$$

the actual spectrum of (50) is

$$E_{n,\mp}(c, \mu_A) = \Omega_B\left(2n \mp \sqrt{\frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B}\mu_A\right)^2} + 1\right) + c,$$

$$n \in \mathbb{N}, \tag{53}$$

where c is a constant term due to the presence of $2\Omega_A \hat{\mathcal{C}}_A$ in Eq. (50) and $\mu_A \approx \hat{J}_{2A}$. Note that when Γ_A is small (in particular $2\Gamma_A\mu_A/\Gamma_B \ll 1/2$), the inverse square potential in (50) can be neglected and the system reduces to that of a QM linear oscillator with a shift term c. This follows also directly from the spectrum (53) provided we set $\Gamma_A = 0$ and consider both signs. In Appendix B we consider the case $m_A = m_B = M$, $\Omega_A = \Omega_B = \Omega$. Then our system reproduces the Smorodinsky–Winternitz system [26] and is related to the Calogero–Moser system [27].

It is interesting to consider the case where μ_A is an imaginary number, i.e., when $\hat{J}_{2A}$ belongs to a non-unitary realization of the D_j series (see Appendix A). Then $R/N > 1/2$, the potential in (51) is repulsive, and the motion takes place only in the domain $r_B \geqslant 0$. This allows to us to view r_B as a radial coordinate and the inverse square potential in (50) and (51) as a rotationally invariant interaction of the spin–orbit type. To see this we rewrite the interaction potential in (50) in the form

$$\hat{H}_{\text{int}} \approx \frac{2}{m_B}\frac{\Gamma_A}{\Gamma_B}\frac{1}{\hat{r}_B}\frac{\partial V}{\partial r_B}(\hat{\mathbf{J}}_B \cdot \hat{\mathbf{J}}_A),$$

$$V = \log r_B. \tag{54}$$

Here we have used (49). We now formally identify $\mathbf{J}_B = i\mathbf{L}$ and $\mathbf{J}_A = i\mathbf{S}$, where $\mathbf{L}$ plays the role of the orbital angular momentum of the "particle" B and $\mathbf{S}$ plays the role of its "spin". The interaction energy then reads

$$H_{\text{int}} = -\frac{2}{m_B}\frac{\Gamma_A}{\Gamma_B}\frac{1}{r_B}\frac{\partial V}{\partial r_B}(\mathbf{L} \cdot \mathbf{S})$$

$$\equiv -\frac{g}{2m_B c^2}\frac{1}{r_B}\frac{\partial V}{\partial r_B}(\mathbf{L} \cdot \mathbf{S}), \tag{55}$$

where we have identified Γ_A/Γ_B with $g/4c^2$, i.e. with a quarter of the gyromagnetic factor (on the second line in (55) we have included Thomas' factor $1/2$). The function V thus plays the role of the planar radial scalar electromagnetic potential. Note that V is the harmonic function in the plane, i.e., $\nabla^2 V(r_B) = \delta(r_B)$.

Eqs. (54) and (55) can be written as

$$H_{\text{int}} = -\mu \cdot \mathbf{B} \tag{56}$$

with

$$\mathbf{B} = \frac{2}{m_B}\frac{1}{\Gamma_B}\frac{1}{r_B}\frac{\partial V}{\partial r_B}\mathbf{J}_B, \tag{57}$$

$$\mu = -\Gamma_A\mathbf{J}_A = -\Gamma_A i\mathbf{S}, \tag{58}$$

thereby suggesting the following interpretation of the emergent global quantum system: In the interaction region (small r_A, r_B) $\mathbf{B}$ plays the rôle of a magnetic field acting on the planar system of magnetic moment μ. $\mathbf{J}_B$ is the orbital angular momentum of the particle "B" and it is orthogonal to the plane where the system lies,

as it is the magnetic field $\mathbf{B}$. Existence of the oscillator "A" is reflected in the constant term contribution $2\Omega_A \hat{\mathcal{C}}_A$ and in the spin of the particle "B" in the spin–orbit interaction term.

We note that also the spin $\mathbf{J}_A \equiv i\mathbf{S}$ is orthogonal to the configuration plane. In conclusion, the emergent global QM system can be interpreted as a particle in a magnetic field, with a spin–orbit interaction term.

Far from the interaction region (large r_A, r_B), the interaction is switched off and the asymptotic emergent QM Hamiltonian reads

$$\hat{H}_T \approx \hat{H}_+$$

$$\approx \left(\frac{\hat{p}_{r_A}^2}{2m_A} + \frac{1}{2}m_A\Omega_A^2\hat{r}_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2}m_B\Omega_B^2\hat{r}_B^2\right). \tag{59}$$

The system is still a genuine QM system because both the two non interacting quantum harmonic oscillators are bounded from below. For small r_A^2, r_B^2 the interaction term is relevant and the global system is no further decomposable into two independent subsystems.

Summarizing, before imposing the information loss condition on the global system, we have two non interacting, independent oscillators. The enforcement of the dissipation constraint on the global system gives rise to the interaction $J_{2A} \leftrightarrow J_{2B}$. So the constraint dictates the form of the interaction. This depends on the dissipations constants Γ_A, Γ_B and can be switched off by setting $\Gamma_A \to 0$. Dissipation thus plays a key rôle in the interaction.

5. Final remarks and conclusions

One can draw an interesting parallel with the conclusions reached in Ref. [28]. There, the authors consider a deformed special relativity model which requires a Planck constant $\hbar$ dependent on the energy scale. In particular, in their model $\hbar(E) \to 0$ for $E \to E_P$, where E is the energy scale of the particle to which the deformed Lorenz boost is to be applied, while E_P is the Planck scale energy. The Planck scale plays in Ref. [28] an analogous rôle as the be-able scale in the 't Hooft case. The basic commutator in [28] is written as

$$[\hat{q}^i, \hat{p}_j^{\text{kin}}] = \delta_j^i\left(1 - \frac{E}{E_P}\right)\hbar, \tag{60}$$

where $\hat{p}_j^{\text{kin}}$ indicates the kinetic momentum operator (i.e., $\hat{p}_j^{\text{kin}} \propto d\hat{q}^j/dt$). In this connection note that the $\hat{p}_i$ introduced in Section 2 is just an auxiliary variable fulfilling $[\hat{q}^i, \hat{p}_j] = \delta_j^i$ at be-able scale, while the $\hat{p}_j^{\text{kin}}$ fulfills at be-able scale $[\hat{q}^i, \hat{p}_j^{\text{kin}}] = 0$. According to (60) the effective Planck constant runs with energy as $\hbar(E) = \hbar(1 - E/E_P)$. For energies $E \ll E_P$ the usual Heisenberg commutator is recovered, but when $E = E_P$ one has $\hbar(E_P) = 0$. Hence in this model the world is classical at the Planck scale, in a similar way as evoked in 't Hooft's proposal, provided one identifies the be-able scale with the Planck scale.

From the viewpoint of the present Letter, the operators $\hat{q}$, $\hat{p}^{\text{kin}}$ should also depend on the energy scale. In fact, at the Planck scale they cannot represent anymore macroscopic concepts as position or momentum, but they should represent be-able degrees of freedom of unknown nature. One should thus write $\hat{q}^i(E/E_P)$ and $\hat{p}_j^{\text{kin}}(E/E_P)$ where, in general, $\hat{q}(0) = \hat{q}$, but $\hat{q}(E/E_P) \neq \hat{q}$ for $0 \ll E \leqslant E_P$ (and analog relations hold for $\hat{p}^{\text{kin}}$). Surely further investigations on the connections between deformed special relativity models and 't Hooft's proposal deserve deeper attention.

Along the line of the previous observation, we can now revisit 't Hooft's constraint ILC (11) (with (12)) by assuming that the "coarse graining" condition scales with the energy (distinguisha-

bility of primordial states degrades with the lowering of energy scale). A simple model that exhibits energy dependence for $\hat{\Phi}$ is

$$\hat{\Phi}_E = \left(1 - e^{-(E_P - \hat{H}_+)\hat{H}_+^{-1}}\right)\hat{H}_-, \tag{61}$$

where E refers to the observer's energy scale, while E_P is the be-able energy scale, which we take to be the Planck energy. An interesting connection with the basic deformed commutator (60) can be established by noting that, from Eq. (60) follows that on physical states,

$$\left(1 - \frac{\hat{H}_+}{E_P}\right) = \frac{1}{n\hbar}\left[\hat{q}^i, \hat{p}_i^{\text{kin}}\right] \tag{62}$$

and therefore $\hat{\Phi}_E$ can be rewritten as

$$\hat{\Phi}_E = \left(1 - e^{-[\hat{q}^i, \hat{p}_i^{\text{kin}}]\hat{H}_+^{-1}E_P/(n\hbar)}\right)\hat{H}_-. \tag{63}$$

Here the rôle of the commutator (60) in defining the coarse-graining operator $\hat{\Phi}_E$ is evident. The operator $\hat{\Phi}_E$ is then, as usual, implemented as a constraint on the be-able Hilbert space $\mathcal{H}$. Therefore at the observational *energy* scale E, the observed *physical* states $|\psi_E\rangle_{\text{phys}}$ must satisfy the condition

$$\hat{\Phi}_E |\psi_E\rangle_{\text{phys}} = 0. \tag{64}$$

This equation identifies the be-able states that are still distinguishable at the observational scale E. The constraint (64) is still a first class primary constraint because $[\hat{\Phi}_E, \hat{\Phi}_E] = 0$ and $[\hat{\Phi}_E, \hat{H}] = 0$. So again it generates gauge transformation [19] and produces equivalence classes of states which are again generally non-local.

Now the group of the gauge transformations generated by $\hat{\Phi}_E$ is a one parameter group $\mathcal{G}_E$. The equivalence classes obtained by such gauge transformations represent at each fixed scale E the physical states (i.e., observables). So the space of the observables will be now denoted by $\mathcal{O}_E$, and identified with the quotient space

$$\mathcal{O}_E = \mathcal{H}_c/\mathcal{G}_E. \tag{65}$$

The quotient space $\mathcal{O}_E$ (its structure and dimensionality) depends on the energy scale E. In particular, at the level of be-ables where $E = E_P$ the constraint $\hat{\Phi}_E$ is identically zero and the space of observables is directly the Hilbert space $\mathcal{H}$. On the other hand, when $E \ll E_P$, e.g., at scales available to an observer, we have $\hat{\Phi} = \hat{H}_-$. The latter is the constraint originally considered by 't Hooft [1,2]. In Section 2 we have dealt precisely with this case.

In conclusion, in this Letter we have considered the problem of quantization of a composite system, in the framework of the quantization scheme proposed by G. 't Hooft [1]. The presented analysis extends the results developed in a previous paper [8], where only a single Bateman's dual oscillator was considered. In this latter case a quantum harmonic oscillator was shown to emerge after the ILC was enforced [8].

We have considered two Bateman's dual oscillators and shown that in this case two possibilities arise: one is to impose the ILC on each Bateman's oscillator separately, thus arriving at two independent quantum harmonic oscillators. Another possibility is to apply the ILC to the composite system of the two Bateman's oscillators, and this in turn leads to an interacting quantum system. We have worked out both possibilities and have shown that the second option leads to an emergent quantum systems which can be identified with the quantum isotonic oscillator [15]. For certain values of parameters this can be interpreted as a particle interacting through a spin–orbit interaction term with an effective magnetic field. Such results are interesting also because they show explicitly that the ILC actually determines the form of the interaction term, and the dissipation controls the interaction strength.

Appendix A

In this appendix we comment more on the physical states $|\psi\rangle_{\text{phys}}$ by closely following Ref. [14]. We denote by $\mathcal{H} = \{|j_i, l_i\rangle, i = A, B\}$, the states corresponding to simultaneous eigenstates of $\hat{J}_{3i}$ and $\hat{C}_i$. These fulfill [22]:

$$\hat{J}_{3i}|j_i, l_i\rangle = \left(l_i + \frac{1}{2}\right)|j_i, l_i\rangle, \tag{66}$$

$$\hat{C}_i|j_i, l_i\rangle = j_i|j_i, l_i\rangle. \tag{67}$$

The corresponding unitary irreducible representations (UIR's) can be grouped into three independent non-overlapping classes (series) according to the spectrum of $\hat{C}_i$ and $\hat{J}_{3i}$. Eigenvalues l_i are in all three series real and discrete. On the other hand, the operators $\hat{C}_i$ and $\hat{J}_{2i}$ generate the so-called non-compact *hyperbolic* subgroup of $su(1,1)$ (see, e.g., Refs. [29,30]). The corresponding UIR's fall into three series. However, the spectrum of $\hat{J}_{2i}$ is either $\mathbb{R} \otimes \mathbb{Z}_2$ or $\mathbb{R}$ [30]. The latter depends on the actual series. Besides UIR's, there exist also (non-unitary) representations in which case $\hat{J}_{2i}$ has discrete complex spectrum. The usual argument that a self-adjoint operator has only real eigenvalues does not apply in this case because here one deals with the extension of $\hat{J}_{2i}$ (that is self-adjoint in a Hilbert space $\mathcal{H}$), to a larger space $\mathcal{D} = \{|\Psi_{j_i, \mu_i}\rangle, i = A, B\}$,

$$\hat{J}_{2i}|\Psi_{j_i, \mu_i}\rangle = \mu_i|\Psi_{j_i, \mu_i}\rangle, \tag{68}$$

$$\hat{C}_i|\Psi_{j_i, \mu_i}\rangle = j_i|\Psi_{j_i, \mu_i}\rangle, \tag{69}$$

in which $\mathcal{H}$ is dense. It can be shown [30] that such an extension is nothing but the closure by continuity of continuous operators defined originally on the dense subspace $\mathcal{H}$.

The choice of a representation corresponding to physical Hilbert space $\mathcal{O}$ is tightly connected with the fact that the state space of the underlying be-able system is $\mathcal{D} \otimes \mathcal{D}$ and that the spectrum of $\hat{H}_i$ should correspond to poles of the resolvent operator $R_z(\hat{H}_i) = (\hat{H}_i - z)^{-1}$. This restricts to a non-unitary realization of the discrete principal series D_j [14,31,32]. The series D_j comes in two copies, namely D_j^+ and D_j^- (where $D_j^- \cap D_j^+ = \{\emptyset\}$). It was shown in Refs. [14,33,34] that, for the Bateman oscillator, D_j^+ corresponds to the forward-time dynamics while D_j^- describes the backward-time dynamics. Denote the *generalized* eigenvectors belonging to $D_j^\pm$ as $|\Psi_{j,\mu}^\pm\rangle$, then [14,30]

$$\hat{J}_{2i}|\Psi_{j_i, \mu_i}^\pm\rangle = \pm\mu_i|\Psi_{j_i, \mu_i}^\pm\rangle, \tag{70}$$

$$\hat{C}_i|\Psi_{j_i, \mu_i}^\pm\rangle = j_i|\Psi_{j_i, \mu_i}^\pm\rangle, \tag{71}$$

with $j_i = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$ and $\mu_i = i(l_i + \frac{1}{2})$, $l_i = |j_i| + m_i$, $m_i \in \mathbb{N}$. There exits a natural outer automorphism between D_j^+ and D_j^- which can be on a physical level identified with the time reversal operation [14,32,33], in particular

$$D_j^+ = \mathcal{T}(D_j^-), \qquad D_j^- = \mathcal{T}(D_j^+), \tag{72}$$

where $\mathcal{T}$ is the time reversal operator.

By imposing the constraint (37) one can identify the physical states $|\psi\rangle_{\text{phys}}$ with states

$$\left\{\left|\psi^+_{j_A,\mu_A}\right\rangle \oplus \left|\psi^-_{j_A,\mu_A}\right\rangle\right\} \otimes \left\{\left|\psi^+_{j_B,\mu_B}\right\rangle \oplus \left|\psi^-_{j_B,\mu_B}\right\rangle\right\}. \tag{73}$$

Here $|\psi^\pm_{j_i,\mu_i}\rangle$ belongs to a two-dimensional space spanned by vectors $\{|\Psi^\pm_{-j_i,\mu_i}\rangle, |\Psi^\pm_{j_i,\mu_i}\rangle\}$.

Appendix B

By considering $m_A = m_B = M$, $\Omega_A = \Omega_B = \Omega$ (this can be achieved by properly rescaling canonical variables and Hamiltonian), we rewrite (50) as

$$\hat{H}_T \approx \hat{H}_+ \approx \frac{\hat{p}^2_{r_A} + \hat{p}^2_{r_B}}{2M} + \frac{1}{2}M\Omega^2\left(\hat{r}^2_A + \hat{r}^2_B\right)$$
$$- \frac{2}{M}\left(\frac{1}{\hat{r}^2_A} + \frac{\Gamma^2_A}{\Gamma^2_B}\frac{1}{\hat{r}^2_B}\right)\hat{J}^2_{2A}, \tag{74}$$

which is known as the two-dimensional Smorodinsky–Winternitz system [26]. It belongs to a class of two-dimensional maximally super-integrable models [35]. Through its centripetal term, the system (74) is related to the Calogero–Moser system [27].

The spectrum of (74) can be written in the form [26]

$$E_{n_A,n_B}(\mu_A) = \Omega\left(2(n_A + n_B) \pm \sqrt{\frac{1}{4} - 4\mu^2_A}\right.$$
$$\left.\pm \sqrt{\frac{1}{4} - \left(\frac{2\mu_A\Gamma_A}{\Gamma_B}\right)^2 + 2}\right), \tag{75}$$

where $n_A, n_B \in \mathbb{N}$. Note that for the attractive potential both signs must be taken into account, while for the repulsive one only "+" sign counts.

When μ_A is a real number in (75), i.e., when $\hat{J}_{2A}$ belongs to a unitary realization of the D_j series, then both potentials in (74) are attractive. As a result the motion takes place in the domain $(r_B, r_A) \in \mathbb{R}^2$ and both signs in (75) must be taken into account. To avoid "fall" to the center [15] both square roots in (75) must be real numbers. This implies that $\mu_A \in [-1/4, 1/4]$ and restricts the possible values of Γ_A/Γ_B to the interval $[-1, 1]$.

References

[1] G. 't Hooft, J. Stat. Phys. 53 (1988) 323;
G. 't Hooft, Class. Quantum Grav. 16 (1999) 3263;
G. 't Hooft, quant-ph/0212095;
G. 't Hooft, Class. Quantum Grav. 13 (1996) 1023.

[2] G. 't Hooft, in: A. Zichichi (Ed.), 37th International School of Subnuclear Physics, Erice, World Scientific, London, 1999, hep-th/0003005;
G. 't Hooft, Int. J. Theor. Phys. 42 (2003) 355;
G. 't Hooft, AIP Conf. Proc. 957 (2007) 154.

[3] T.S. Biro, S.G. Matinyan, B. Muller, Found. Phys. Lett. 14 (2001) 471.

[4] R. Banerjee, Mod. Phys. Lett. A 17 (2002) 631.

[5] H.-T. Elze, J. Phys.: Conf. Ser. 33 (2006) 399.

[6] H.-T. Elze, Int. J. Quantum Info. (IJQI) 7 (2009) 83.

[7] J.J. Halliwell, Phys. Rev. D 63 (2001) 085013.

[8] M. Blasone, P. Jizba, G. Vitiello, Phys. Lett. A 287 (2001) 205.

[9] M. Blasone, P. Jizba, G. Vitiello, J. Phys. Soc. Jap. Suppl. 72 (2003) 50;
M. Blasone, P. Jizba, Can. J. Phys. 80 (2002) 645;
M. Blasone, E. Celeghini, P. Jizba, G. Vitiello, Phys. Lett. A 310 (2003) 393.

[10] F. Scardigli, Found. of Phys. 37 (2007) 1278.

[11] M. Blasone, P. Jizba, H. Kleinert, Phys. Rev. A 71 (2005) 052507.

[12] M. Blasone, P. Jizba, H. Kleinert, Ann. Phys. 320 (2005) 468.

[13] M. Blasone, P. Jizba, J. Phys.: Conf. Ser. 67 (2007) 012046.

[14] E. Celeghini, M. Rasetti, G. Vitiello, Ann. Phys. 215 (1992) 156.

[15] L.D. Landau, E.M. Lifshitz, Quantum Mechanics: Non-relativistic Theory, Pergamon, New York, 1977.

[16] L.S. Pontryagin, V.G. Boltanskij, R.V. Gamkrelidze, E.F. Miscenko, The Mathematical Theory of Optimal Processes, Wiley, New York, 1962.

[17] Hermiticity of $\hat{H}$ is not a problem at the energy scale E_P, since the Hamiltonian is only a formal tool useful to generate evolution equations. However, one can always compensate for non-hermitian ordering by adding the non dynamical arbitrary function $g(\hat{\mathbf{q}})$ in such a way that $g^\dagger(\hat{\mathbf{q}}) - g(\hat{\mathbf{q}}) = \sum_i[\hat{p}_i, f_i(\hat{\mathbf{q}})]$ so that $\hat{H} = \hat{H}^\dagger$

[18] J.S. Bell, Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press, Cambridge, 1987.

[19] See, e.g., K. Sundermeyer, Constrained Dynamics with Applications to Yang–Mills Theory, General Relativity, Classical Spin, Dual String Model, Springer Verlag, Berlin, 1982;
M. Henneaux, C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, Princeton, 1991;
P.A.M. Dirac, Can. J. Math. 2 (1950) 129;
P.A.M. Dirac, Proc. R. Soc. London A 246 (1958) 326;
P.A.M. Dirac, Lectures in Quantum Mechanics, Yeshiva University, New York, 1965.

[20] H. Umezawa, G. Vitiello, Quantum Mechanics, Bibliopolis, Napoli, 1985.

[21] M. Blasone, E. Graziano, O.K. Pashaev, G. Vitiello, Ann. Phys. 252 (1996) 115.

[22] A. Perelomov, Generalized Coherent States and Their Applications, Springer-Verlag, London, 1986.

[23] J.S. Wang, T.K. Liu, M.S. Zhan, J. Optics B 2 (2000) 758.

[24] K. Thirulogasanthar, N. Saad, J. Phys. A 37 (2004) 4567.

[25] Z. Dongpei, J. Phys. A.: Math. Gen. 20 (1987) 4331.

[26] C. Grosche, G.S. Pogosyan, A.N. Sissakian, Fortschr. Phys. 43 (1995) 6.

[27] See, e.g., E. Corrigan, R. Sasaki, J. Phys. A 35 (2002) 7017, and citations therein.

[28] J. Magueijo, L. Smolin, Phys. Rev. D 67 (2003) 044017.

[29] A. Barut, R. Raczka, Theory of Group Representations and Applications, Polish Scientific Publisher, Warszawa, 1977.

[30] G. Lindblad, B. Nagel, Ann. Inst. Henri Poincaré Sect. A 13 (1970) 27.

[31] H. Feshbach, Y. Tikochinski, Trans. N.Y. Acad. Sci., Ser. II 38 (1977) 44.

[32] D. Chruscinski, Ann. Phys. 321 (2006) 840.

[33] M. Blasone, P. Jizba, Ann. Phys. 312 (2004) 354.

[34] D. Chruscinski, J. Jurkowski, Ann. Phys. 321 (2006) 856.

[35] J. Friš, V. Mandrosov, Ya.A. Smorodinsky, M. Uhlíř, P. Winternitz, Phys. Lett. 16 (1965) 354.