Localized oscillating configurations formed by real scalar fields

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Long-living regular localized solutions in general relativity

**Geons** formed by gravitational waves
- very high frequency waves needed – averaging methods
- lifetime may be short

**Scalar fields** coupled to gravity
- spherically symmetric solutions exist
- for massive scalar field no need for averaging

**Negative cosmological constant** – solutions without averaging
- vacuum geons
- scalar field can be massless

Most of the talk is about $\Lambda = 0$ scalars.
at the end short summary of results for $\Lambda < 0$
Gravitational attraction forms a spherically symmetric star-like object from a scalar field

for a complex scalar field it is called **boson star**

for a real scalar field it is called **oscillaton**

Extremely long living and stable, but for $\Lambda = 0$ the mass of oscillatons decreases very slowly because of a tiny scalar field radiation
Discovery of oscillatons (oscillating soliton stars)

Time-evolution of a real Klein-Gordon field coupled to gravity


\[ \text{time evolution of } r^2 \rho, \text{ where } \rho \text{ is the energy density} \]

get rid of the unnecessary energy by “gravitational cooling”

a large class of spherically symmetric initial data evolve into oscillatons
Up to now there are only about 30 papers published on oscillatons:
\[ \sim 500 \text{ about boson stars} \]
- boson stars do not radiate
- metric of boson stars is static

**Oscillatons necessarily radiate**

- results for classical and quantum radiation
- for small mass states quantum decay dominate

**3 + 1 dimensional time-evolution code:**

- nonspherical perturbations die off in a short timescale, emitting gravitational waves
Exactly periodic but only weakly localized solutions

Energy loss is compensated by small incoming radiation in order to make the solution exactly time-periodic.

There are time-periodic solutions with very low energy density standing wave tails—called nanopterons by J. P. Boyd (Nonlinearity 3, 177, 1990).
Quasibreathers

The exactly periodic solution with the minimal energy density radiative tail is named **quasibreather**

Quasibreathers agree to a high precision with the same frequency oscillaton in the core region and have the same tail amplitude.

**Oscillatons evolve adiabatically through quasibreather states**

Quasibreathers are much easier to study numerically – system of ordinary differential equations.

**Total mass of quasibreathers is infinite** – energy density of tail is small, but have to integrate it to infinite volume.

Quasibreather states are not asymptotically flat.
Massive real Klein-Gordon field coupled to gravity

Field equations:

\[ g^{ab} \phi_{,ab} - m^2 \phi = 0 \]

\[ G_{ab} = T_{ab} \]

where the stress-energy tensor is

\[ T_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} (\phi_{,c} \phi^{,c} + m^2 \phi^2) \]

We use isotropic coordinates:

\[ ds^2 = -Adt^2 + B \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

– rescaling \( t \) and \( r \) we set the scalar field mass to \( m = 1 \)
Fourier decomposition

of the field $\phi$ and metric components $A = -g_{tt}$ and $B = g_{rr}$

$$\phi = \phi_1 \cos(\omega t) + \phi_3 \cos(3\omega t) + \phi_5 \cos(5\omega t) + \ldots$$

$$A = 1 + A_0 + A_2 \cos(2\omega t) + A_4 \cos(4\omega t) + \ldots$$

$$B = 1 + B_0 + B_2 \cos(2\omega t) + B_4 \cos(4\omega t) + \ldots$$

where $\phi_k$, $A_k$ and $B_k$ depend only on $r$

Getting the standing wave tail requires very high precision

– we use the spectral library Kadath of Philippe Grandclément

– none of the earlier numerical calculations were precise enough to find the tail
Boundary conditions

Regularity at the center

\[ \frac{d\phi_n}{dr} \bigg|_{r=0} = 0, \quad \frac{dA_n}{dr} \bigg|_{r=0} = 0, \quad \frac{dB_n}{dr} \bigg|_{r=0} = 0 \]

The asymptotic large \( r \) behavior for \( n = 1 \) is decaying

\[ \phi_1 = \frac{\alpha_1}{r} \exp \left(-r \sqrt{1 - \omega^2} \right) \]

for \( n = 3, 5, 7 \ldots \) is oscillating

\[ \phi_n = \frac{\alpha_n}{r} \sin \left( r \sqrt{n^2\omega^2 - 1} \right) + \frac{\beta_n}{r} \cos \left( r \sqrt{n^2\omega^2 - 1} \right) \]

\( \alpha_n = \beta_n = 0 \) and regularity are too many boundary conditions

\[ \rightarrow \text{ no exactly periodic localized oscillaton is expected} \]

There is a one-parameter family of (nodeless) oscillatons

Time-averaged central density is a monotonically increasing function of the central amplitude
– for small amplitudes it is proportional to $\phi_c^2$

In general, a star-like astrophysical object is stable if the total mass increases with increasing central density

Oscillatons with amplitude larger then a certain value are unstable
Radius $R_{10}$ where amplitude decreases to $1/10$ of central amplitude

for small amplitudes $\sim \frac{1}{\sqrt{\phi_c}}$

As the central amplitude decreases, the frequency approaches $m = 1$ from below

Radiation decreases exponentially with the decreasing amplitude → oscillatons have infinite lifetime
Oscillatons oscillate almost periodically, with an extremely slowly increasing frequency

Behavior of the scalar field during one oscillation period for the largest amplitude stable oscillaton

Scalar field decreases exponentially up to \( r = 32 \), where a radiative tail begins, with amplitude \( 10^{-8} \)

For smaller amplitude oscillatons the tail is even much smaller – the tail decreases exponentially with decreasing central amplitude
Metric components oscillate around a static background

For smaller amplitude configurations the relative oscillation of the metric is even smaller

For boson stars, formed by a complex scalar field, the metric is static
Absolute value of the Fourier components of $\phi$ for the highest amplitude stable oscillation ($\omega = 0.8608$)

- the downward spikes correspond to zero crossings
- for lower amplitude oscillations the higher modes are even more suppressed
The standing wave tail is dominated by the \( \cos(3\omega t) \) mode

\[
\phi_3 = \frac{C_3}{r} \sin \left( r \sqrt{9\omega^2 - 1 + \delta} \right)
\]

Asymptotic series representation of the core region

The small parameter can be chosen \( \varepsilon = \sqrt{1 - \omega^2} \)

To leading order

– the amplitude is proportional to \( \varepsilon^2 \)
– the size is proportional to \( 1/\varepsilon \)


The standing or outgoing wave tail is exponentially small in \( \varepsilon \), so it cannot be obtained directly by the small-amplitude expansion
Small-amplitude expansion

We use isotropic coordinates

\[ ds^2 = -B(t, r)dt^2 + \frac{W(t, r)}{B(t, r)} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \]

\( \varepsilon \) expansion of the scalar field and metric functions:

\[ \phi = \sum_{n=1}^{\infty} \varepsilon^{2n} \phi_{2n} \, , \quad B = 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} B_{2n} \, , \quad W = 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} W_{2n} \]

Since for small amplitudes the size is proportional to \( 1/\varepsilon \), use \( \rho = \varepsilon r \) as the radial coordinate

It turns out that to leading order metric is static

\[ \phi_2(t, \rho) = p(\rho) \cos(t) \, , \quad B_2(t, \rho) = b(\rho) \, , \quad W_2(t, \rho) = 0 \]
To leading order oscillatons are described by the Schrödinger-Newton equations

\[ \frac{d^2}{d\rho^2}(\rho p) = \rho bp \quad , \quad \frac{d^2}{d\rho^2}(\rho b) = \frac{1}{2}\rho p^2 \]

\(b\) and \(p\) are functions of \(\rho\)

SN equations also describe the Newtonian limit of boson stars
Nodeless solution of Schrödinger-Newton equations

\[ \phi(\tau, r) = \varepsilon^2 p(\varepsilon r) \cos(\omega \tau) \]

\[ ds^2 = -\omega^2 B(r) d\tau^2 + \frac{1}{B(r)} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \]

where \( B(r) = 1 + \varepsilon^2 b(\varepsilon r) \) and \( \omega = \sqrt{1 - \varepsilon^2} \)

– the oscillation of the metric begins at \( \varepsilon^4 \) order
Method of Segur and Kruskal

Originally for $1 + 1$ dimensional oscillons (real scalar, flat background)


Extend both the $\varepsilon$ expansion and Fourier mode equations to the complex $r$ plane, and study their behavior near the pole on the imaginary axis.

The solutions have pole singularities on the imaginary axis at

$$r = \pm \frac{iQ}{\varepsilon}$$

corresponding to the poles of the Schrödinger-Newton equations at

$$\rho = \pm iQ \quad (Q \approx 3.977).$$

As $\varepsilon \to 0$ the poles move away from the real axis, but close to the poles $\bar{\phi}_k$, $\bar{A}_k$ and $\bar{A}_k$ have $\varepsilon$ independent parts.

– on the real axis everything goes to zero in the $\varepsilon \to 0$ limit.
Correction to the asymptotic series

Near the pole there is an exponentially small correction to the result of the small-amplitude expansion
– this corresponds to the missing oscillating tail

Since on the imaginary axis the $\varepsilon$ expansion gives real functions, near the pole “beyond all order” corrections can be calculated to the imaginary parts
– these corrections decay exponentially towards the real axis

Segur and Kruskal used numerical integration to calculate them

They can also be calculated analytically using Borel summation
Continuation back to the real axis

Once these corrections are known near the pole, we can continue them down to the real axis, at distance $Q/\varepsilon$. The correction becomes exponentially small in $\varepsilon$.

Continuing along the real axis to the asymptotic region, this will provide the oscillating tail responsible for the energy loss.


Tail amplitude:

$$C_3 = -\frac{3.76}{\varepsilon} \exp\left(-\frac{11.25}{\varepsilon}\right)$$

The tail oscillates with frequency $3\omega t$, where $\omega = \sqrt{1 - \varepsilon^2}$.
Mass loss rate

We have to calculate the mass-energy carried by the outgoing spherical scalar wave

$$\Phi = \frac{C_3}{r} \cos \left( r \sqrt{9\omega^2 - 1} - 3\omega t \right)$$

Taking the large $r$ limit, and averaging in time, the mass loss rate is proportional to $C_3^2$

$$\frac{dM}{dt} = -\frac{30.0}{\varepsilon^2} \exp \left( -\frac{22.4993}{\varepsilon} \right)$$

The $\varepsilon$ dependence of the total mass for small amplitudes is

$$M = \frac{1}{m} \left[ 1.753 \varepsilon - 2.117 \varepsilon^3 \right]$$

$\varepsilon$ slowly changes with time
Fraction of the mass lost during one oscillation period

\[ \frac{\Delta M}{M} \] is independent of the scalar field mass \( m \)
The mass of an initially maximal mass oscillaton as a function of time

The elapsed time $t$ is inversely proportional to the scalar field mass $m$, so we plot the product $tm$.
Mass lost since the early universe

Start with a maximal mass oscillaton 13.7 billion years ago, choosing various scalar field masses $m$ we list how big part of the mass is lost by now

<table>
<thead>
<tr>
<th>$m \frac{c^2}{eV}$</th>
<th>$\frac{M_{\text{max}} - M}{M_{\text{max}}}$</th>
<th>$\frac{M}{M_{\odot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-25}$</td>
<td>$3.16 \cdot 10^{-5}$</td>
<td>$8.09 \cdot 10^{14}$</td>
</tr>
<tr>
<td>$10^{-15}$</td>
<td>$0.0896$</td>
<td>$7.36 \cdot 10^4$</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>$0.263$</td>
<td>$5.96 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$0.401$</td>
<td>$4.85 \cdot 10^{-16}$</td>
</tr>
<tr>
<td>$10^{15}$</td>
<td>$0.500$</td>
<td>$4.04 \cdot 10^{-26}$</td>
</tr>
</tbody>
</table>

the last column gives the resulting mass $M$ in solar mass units

Negative cosmological constant

There are exactly periodic oscillatons without radiative tail
better name: AdS Breathers
– minimally coupled massless Klein-Gordon field is enough
– size is determined by the cosmological constant $\sim 1/\sqrt{-\Lambda}$


There is a one-parameter family of nodeless solutions
Mass as function of the oscillation frequency

AdS breather becomes unstable when the total mass starts to decrease with increasing central density.

First two orders of the small-amplitude expansion is also plotted.