Emergent Spacetime for Quantum Gravity

Hyun Seok Yang

Center for Quantum Spacetime
Sogang University

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In this talk

- I emphasize that noncommutative (NC) spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC.
- The emergent gravity from NC $U(1)$ gauge theory is the large $N$ duality and the emergent spacetime picture admits a background-independent formulation of quantum gravity.
- In order to understand NC spacetime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years.
References


H. S. Yang, Emergent spacetime and cosmic inflation, arXiv: 1503.00712

And references therein.
NC spacetime introduces the gauge-gravity duality

1. Recall that quantum mechanics is mechanics on NC phase space whose coordinate generators satisfy the commutation relation

\[ [x^i, p_j] = i\hbar \delta_j^i \]

2. The mathematical structure of NC spacetime is essentially the same as the NC phase space in quantum mechanics:

\[ [y^\mu, y^\nu] = i\theta^{\mu\nu}, \]

where \( p_\mu = B_{\mu\nu} y^\nu \) and \( B_{\mu\nu} \equiv (\theta^{-1})_{\mu\nu}. \)

3. Everything on NC spacetime bears some analogy with quantum mechanics:

NC phase space \( \Rightarrow \) Wave-particle duality

NC spacetime \( \Rightarrow \) Gauge-gravity duality
NC spacetime necessarily implies emergent spacetime

1. Recall that \( f(x + a) = U(a)^\dagger f(x) U(a) \) where \( U(a) = e^{-i p \cdot a / \hbar} \), and so every point on NC space is unitarily equivalent. There is no space but an algebra \( \mathcal{A}_\theta \) only. Thus the NC space is a misnomer.

2. NC space introduces a separable Hilbert space \( \mathcal{H} \) and so \( \mathcal{H} \) has a countable basis. Dynamical variables become operators acting on the Hilbert space.

3. NC spacetime implies a paradigm shift: Geometry \( \rightarrow \) Algebra
   Hilbert space \( \mathcal{H} \): dynamical variables \( \rightarrow \) \( N \times N \) matrices where \( N = \dim(\mathcal{H}) \rightarrow \infty \).

4. Large \( N \) duality or gauge/gravity duality such as the AdS/CFT correspondence is an inevitable consequence of the NC spacetime.

5. NC spacetime admits a (dynamical) diffeomorphism symmetry which precisely acts as the novel form of the equivalence principle for electromagnetic force.
NC Fields As Large N Matrices

Consider a two-dimensional NC space

\[ [x, y] = i \theta \quad \iff \quad [a, a^\dagger] = 1 \text{ where } a = \frac{x + iy}{\sqrt{2\theta}}. \]

Since \( \mathcal{H} = \{ |n\rangle ; n = 0, 1, \cdots, \infty \} \) and \( \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}_\mathcal{H} \), for \( \phi_1, \phi_2 \in \mathcal{A}_\theta \),

\[ \phi_1(x, y) = \sum_{n,m=0}^{\infty} |n\rangle\langle n| \phi_1(x, y) |m\rangle\langle m| \equiv M_{nm} |n\rangle\langle m|, \]

\[ \phi_2(x, y) = \sum_{n,m=0}^{\infty} |n\rangle\langle n| \phi_2(x, y) |m\rangle\langle m| \equiv N_{nm} |n\rangle\langle m|, \]

\( (\phi_1 \ast \phi_2)(x, y) = \sum_{n,l,m=0}^{\infty} |n\rangle\langle n| \phi_1(x, y) |l\rangle\langle l| \phi_2(x, y) |m\rangle\langle m| = M_{nl} N_{lm} |n\rangle\langle m|, \)

NC fields \( \phi_a(x, y) \) in \( \mathcal{A}_\theta = \text{adjoint operators acting on a separable} \)
Hilbert space \( \mathcal{H} = N \times N \) matrices in \( \text{End}(\mathcal{H}) \equiv \mathcal{A}_N \) with \( N = \text{dim}(\mathcal{H}) \to \infty. \)

Ordering in \( \mathcal{A}_\theta = \text{ordering in } \mathcal{A}_N \) and \( \text{Tr}_N = \text{Tr}_\mathcal{H} = \int \frac{dx \, dy}{2\pi \theta}. \)
Figure 1: Flowchart for emergent gravity
Large N gauge theory from NC U(1) gauge theory

Consider a \((d+2n)\)-dimensional NC U(1) gauge theory on \(\mathbb{R}^d \times \mathbb{R}^{2n}_{NC}\) whose coordinates are \(X^M = (x^\mu, y^a)\), \(M = 0, 1, \ldots, D - 1\), \(\mu = 0, 1, \ldots, d - 1\), \(a = 1, \ldots, 2n\) where 
\[ [y^a, y^b] = i\theta^{ab}. \]

The \(D=(d+2n)\)-dimensional U(1) connections are split as
\[ D_M(X) = \partial_M - i\hat{A}_M(x, y) = (D_\mu, D_a)(x, y) \]
where \(\partial_a \equiv \text{ad}_{p_a} = -i[p_a, \cdot] \) with \(p_a = B_{ab} y^b\) and
\[ D_a(x, y) = -i(p_a + \hat{A}_a(x, y)) \equiv -i\phi_a(x, y). \]

Using the matrix representation \(\mathcal{A}_\theta \rightarrow \mathcal{A}_N\) by
\[ \Xi(x, y) \rightarrow \Xi(x) \in \mathcal{A}_N, \]
the D-dimensional NC U(1) gauge theory is exactly mapped to the d-dimentional U(N) Yang-Mills theory
\[ S = -\frac{1}{4g_{YM}^2} \int d^D X \left( \hat{F}_{MN} - B_{MN} \right)^2 \]
\[ = -\frac{1}{g_{YM}^2} \int d^dx \text{Tr}(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_{\mu} \phi_a D^{\mu} \phi_a - \frac{1}{4} [\phi_a, \phi_b]^2) \]

where \(B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \).

\(d=0\): IKKT, \(d=1\): BFSS, \(d=2\): DVV, \(d=4\): AdS/CFT, \(d=5\): etc.
NC spacetime as NC Coulomb branch

I will make an important observation that NC spacetime arises as a vacuum solution in the Coulomb branch of a large N gauge theory to demonstrate the large N duality.

1. The conventional choice of vacuum in the Coulomb branch of $U(N)$ Yang-Mills theory is given by

$$[\phi_a, \phi_b]_{\text{vac}} = 0 \quad \Rightarrow \quad \langle \phi_a \rangle_{\text{vac}} = \text{diag}((\alpha_a)_1, (\alpha_a)_2, \cdots, (\alpha_a)_N)$$

In this case the $U(N)$ gauge symmetry is broken to $U(1)^N$.

2. If we consider the $N \to \infty$ limit, the large N limit opens a new phase of the Coulomb branch given by

$$[\phi_a, \phi_b]_{\text{vac}} = -i B_{ab} \quad \Rightarrow \quad \langle \phi_a \rangle_{\text{vac}} = p_a \equiv B_{ab} y^b$$

where the vacuum moduli $y^a \in \mathbb{R}_{NC}^{2n}$ satisfy the Moyal-Heisenberg algebra.

3. Suppose that fluctuations around the NC vacuum take the form

$$D_\mu = \partial_\mu - i \hat{A}_\mu(x, y), \quad \phi_a = p_a + \hat{A}_a(x, y)$$

The adjoint scalar fields now obey the deformed algebra given by

$$[\phi_a, \phi_b] = -i (B_{ab} - \hat{F}_{ab})$$
Large N duality from NC spacetime

Plugging the fluctuations into the \(d\)-dimensional \(U(N \rightarrow \infty)\) Yang-Mills theory, we get the \(D = (d + 2n)\)-dimensional NC \(U(1)\) gauge theory and thus arrive at the reversed version of the equivalence

\[
S = -\frac{1}{g^2_{YM}} \int d^d x \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a - \frac{1}{4} [\phi_a, \phi_b]^2 \right) \\
= -\frac{1}{4G^2_{YM}} \int d^D X \left( \hat{F}_{MN} - B_{MN} \right)^2
\]

where \(X^M = (x^\mu, y^a)\) are D-dimensional coordinates and D-dimensional connections are defined by

\[
D_M(X) = \partial_M - i \hat{A}_M(x, y) = (D_\mu, D_a)(x, y)
\]

whose field strength is given by

\[
\hat{F}_{MN}(X) = \partial_M \hat{A}_N - \partial_N \hat{A}_M - i[\hat{A}_M, \hat{A}_N]_x.
\]
\[ U(N \to \infty) \text{ Yang-Mills theory on } \mathbb{R}^{d-1,1} \]

NC Coulomb branch \hspace{2cm} \text{Large } N \text{ duality}

\[ \text{NC } U(1) \text{ gauge theory on } \mathbb{R}^{d-1,1} \times \mathbb{R}_\theta^{2n} \]

Inner derivation \hspace{2cm} \text{Classical limit}

\[ D = d + 2n \text{-dimensional Einstein gravity} \]

Differential operators as quantized frame bundle

Figure 2: Flowchart for large } N \text{ duality
Inner/Outer automorphism

Therefore there should be some way to map the NC $U(1)$ gauge theory to the Einstein gravity according to the (conjectural) large $N$ duality. To be more specific, consider the inverse metric in Einstein gravity given by

$$
\left( \frac{\partial}{\partial s} \right)^2 = E_A \otimes E_A = g^{MN}(X) \partial_M \otimes \partial_N
$$

where $E_A = E^M_A(X) \partial_M$ are orthonormal frames on the tangent bundle $TM$ of a $D$-dimensional spacetime manifold $\mathcal{M}$. In order to complete the gauge-gravity duality in Figs. 1 and 2, it is thus necessary to realize the vector fields $E_A = E^M_A(X) \partial_M \in \Gamma(T\mathcal{M})$ in terms of NC $U(1)$ gauge fields.

A decisive clue is coming from the fact that the NC $\star$-algebra $\mathcal{A}_\theta$ generated by the Moyal-Heisenberg algebra always admits a nontrivial inner automorphism $\mathcal{I}$. The infinitesimal generators of $\mathcal{I}$ form an inner derivation defined by the adjoint operation

$$
\mathcal{A}^d_\theta \rightarrow \mathcal{D}^d : f \mapsto \text{ad}_f = -i[f, \cdot]
$$

for any $f \in \mathcal{A}^d_\theta \equiv \mathcal{A}_\theta(C^\infty(\mathbb{R}^{d-1,1})) \cong \mathcal{A}_\theta \times C^\infty(\mathbb{R}^{d-1,1})$

Definitely the derivation $\mathcal{D}^d$ is a Lie algebra homomorphism, i.e.,

$$
\text{ad}_{[f,g]} = i[\text{ad}_f, \text{ad}_g]
$$
**Vielbeins from inner derivation**

Consider the derivation algebra generated by the dynamical variables defined by

\[ \hat{V}_A = \{i \text{ad}_{D_A} = [D_A, \cdots ]|D_A(x, y) = (D_\mu, D_a)(x, y)\} \in \mathcal{D}^d \]

where \( D_A(x, y) = -i \phi_A(x, y) \). In a large-distance limit, i.e. \(|\theta| \to 0\), one can expand the NC vector fields using the explicit form of the Moyal \( \star \)-product. The result takes the form

\[ \hat{V}_A = V_A^M(x, y) \frac{\partial}{\partial Y_M} + \sum_{p=2}^{\infty} V_A^{a_1\cdots a_p}(x, y) \frac{\partial}{\partial y^{a_1}} \cdots \frac{\partial}{\partial y^{a_p}} \in \mathcal{D}^d, \]

where \( V_A^\mu = \delta_A^\mu \). Thus the Taylor expansion of NC vector fields generates an infinite tower of the so-called polyvector fields. Note that the leading term gives rise to the ordinary vector fields that will be identified with a frame basis associated with the tangent bundle \( TM \) of an emergent spacetime manifold \( \mathcal{M} \).

Let us truncate the above polyvector fields to ordinary vector fields given by

\[ \mathfrak{x}(\mathcal{M}) = \left\{ V_A = V_A^M(x, y) \frac{\partial}{\partial X_M} | A, M = 0, 1, \ldots, D - 1 \right\} \]

where \( X^M = (x^\mu, y^a) \) are local coordinates on a \( D \)-dimensional emergent Lorentzian manifold \( \mathcal{M} \).
Emergent gravity from NC spacetime

The orthonormal vielbeins on $TM$ are then defined by the relation

$$V_A = \lambda E_A \in \Gamma(TM)$$

or on $T^*M$

$$\nu^A = \lambda^{-1} e^A \in \Gamma(T^*M).$$

The conformal factor $\lambda \in C^\infty(M)$ is determined by the volume preserving condition

$$\mathcal{L}_{V_A} \nu_t = (\nabla \cdot V_A + (d - 2n) V_A \ln \lambda) \nu_t = 0, \quad \forall A = 0, 1, \ldots, D - 1,$$

where

$$\nu_t \equiv d^d x \wedge \nu = \lambda^2 d^d x \wedge v^1 \wedge \cdots \wedge v^{2n}$$

is a $D$-dimensional volume form on $M$.

In the end, the Lorentzian metric on a $D$-dimensional spacetime manifold is given by

$$ds^2 = g_{MN}(X) dX^M \otimes dX^N = e^A \otimes e^A$$

$$= \lambda^2 \nu^A \otimes \nu^A = \lambda^2 \left( \eta_{\mu\nu} dx^\mu dx^\nu + v^a_b v^a_c (dy^b - A^b) (dy^c - A^c) \right)$$

where $A^b := A^b_\mu(x, y) dx^\mu$. Therefore the NC field theory representation of the $d$-dimensional large $N$ gauge theory in the NC Coulomb branch provides a powerful machinery to identify gravitational variables dual to large $N$ matrices.
Line bundle and D-brane

A line bundle over $M$ can be modeled by the open string theory on a D-brane whose worldvolume is a symplectic manifold ($M = \mathbb{C}^n, B$).

Low energy effective action of a D-brane: DBI action

$$S = \frac{1}{g_s} \int d^{p+1}x \sqrt{\det(g + \kappa (B + F))}$$

Bundle isomorphism: $B: TM \to T^*M$ by $X \mapsto A = \iota_X B$.
Then $F = dA = \mathcal{L}_XB$ where $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to $X$.

Dynamical Diff ($M$): $\mathcal{F} = B + F = (1 + \mathcal{L}_X)B \approx e^{\mathcal{L}_X}B$

Note that a vector field is an infinitesimal generator of a local coordinate transformation, i.e., a Lie algebra generator of Diff ($M$).

Darboux theorem: $\exists \phi \in$ Diff ($M$) such that $\phi^*(B + F) = B$ where $\phi^* = (1 + \mathcal{L}_X)^{-1} \approx e^{-\mathcal{L}_X}$: exponential map
**Darboux theorem**

**Equivalence principle for electromagnetic force:** The Darboux theorem states that it is possible to find a local coordinate transformation $\phi \in \text{Diff} (M)$ which eliminates dynamical $U(1)$ gauge fields in $\mathcal{F}$.

In terms of local coordinates, $\exists \phi: y \mapsto x = x(y)$ such that

$$(B_{ab} + F_{ab}(x)) \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = B_{\mu\nu}$$

If we represent the local coordinate transformation by $x^a(y) = y^a + \theta^{ab} \hat{A}_b(y)$ with $\theta \equiv B^{-1}$,

$$\Theta^{ab}(x) \equiv (\mathcal{F}^{-1})^{ab}(x) = \{x^a(y), x^b(y)\}_\theta = [\theta(B - \hat{F})\theta]^{ab}(y)$$

where the Poisson bracket for $f, g \in C^\infty(M)$ is defined by

$$\{f(y), g(y)\}_\theta = \theta^{\mu\nu} \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu}$$

and

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \{\hat{A}_\mu, \hat{A}_\nu\}_\theta$$
Seiberg-Witten map

From the Darboux transformation, we get

\[ \hat{F}_{\mu\nu}(y) = \left( \frac{1}{1 + F \theta} F \right)^{\mu\nu}(x), \]  

\[ d^{2n} y = d^{2n} x \sqrt{\det(1 + F \theta)} \]

and

\[ g_{ab} + \kappa F_{ab}(x) = (g_{\mu\nu}(y) + \kappa B_{\mu\nu}) \frac{\partial y^\mu}{\partial x^a} \frac{\partial y^\nu}{\partial x^b} \]

where the dynamical (emergent) metric is defined by

\[ g_{\mu\nu}(y) = g_{ab} \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} \]
The Seiberg-Witten map in turn leads to a remarkable identity between DBI actions

\[
\frac{1}{g_s} \int d^{2n}x \sqrt{\det (g + \kappa F)} = \frac{1}{g_s} \int d^{2n}y \sqrt{\det (g + \kappa B)}
\]

\[
= \frac{1}{g_s} \int d^{2n}y \sqrt{\det (G + \kappa (\hat{F} + \Phi))}
\]

where the open and closed string parameters are related by

\[
\frac{1}{g + \kappa B} = \frac{1}{G + \kappa \Phi} + \frac{\theta}{\kappa}
\]

\[
G_s = g_s \sqrt{\frac{\det (G + \kappa \Phi)}{\det (g + \kappa B)}}
\]

Up to total derivatives, we may have a local form of the equivalence

\[
\sqrt{\det (g + \kappa (B + F))} = \sqrt{\det (g + \kappa B)} \quad \text{(A)}
\]

\[
= \frac{g_s}{G_s} \sqrt{\det (G + \kappa (\hat{F} - B))} \quad \text{(B)}
\]
Kähler manifolds from $U(1)$ gauge fields

Suppose that $M = \mathbb{C}^n$. For our case ($\kappa = 1$),

\[ g_{\mu\nu}, G_{\mu\nu} : \text{Kähler metrics of } \mathbb{C}^n, \text{i.e., } g_{ij} = G_{ij} = \delta_{ij} = \partial_i \bar{\partial}_j K_0 \text{ where } K_0 = z^k \bar{z}^k \]

\[ B_{\mu\nu} = -2I_{\mu\nu} : \text{Kähler form of } \mathbb{C}^n, \text{i.e., } B = -i \partial \bar{\partial} K_0 \]

Now we want to find $U(1)$ gauge fields that give rise to a Kähler metric

\[ g_{ij} = \partial_i \bar{\partial}_j K(z, \bar{z}) \]

This means that the RHS of Eq. (A) is purely of $(1,1)$-type. Therefore, to satisfy Eq. (A), the $U(1)$ gauge fields on the LHS must be a connection of a holomorphic line bundle obeying $\mathcal{F}_{ij} = \mathcal{F}_{i\bar{j}} = 0$. Then we have

**LHS:**

\[ g_{ij} + \mathcal{F}_{ij} = i \partial_i \bar{\partial}_j (\phi(z, \bar{z}) - i K_0) \]

\[ \implies \phi(z, \bar{z}) = K(z, \bar{z}). \]

**RHS:**

\[ g_{i\bar{j}} + B_{i\bar{j}} = \partial_i \bar{\partial}_j (K(z, \bar{z}) - i K_0) \]

up to holomorphic gauge transformations.
Quantum foam and topological strings

A holomorphic line bundle with a nondegenerate curvature two-form of rank $2n$ is equivalent to a $2n$-dimensional Kähler manifold.

This fact was beautifully used to realize the Kähler gravity in terms of U(1) gauge theory in the paper, "Quantum foam and topological strings" by Amer Iqbal, Cumrun Vafa, Nikita Nekrasov and Andrei Okounkov published in JHEP 04 (2008) 011, hep-th/0312022:

(1. Introduction) “Thus we conclude that for topological strings a gauge theory is the fundamental description of gravity at all scales including the Planck scale, where it leads to a quantum gravitational foam.”

(6. Target space theory viewpoint) “The aim of this section is to propose a precise (gauge) theory which plays the role of target space Kaehler gravity. Just like the three (and two) dimensional gravities have gauge formulations (involving SL2,E(2), etc.), Kaehler gravity in any dimension is in some sense a U(1) gauge theory.”

Unfortunately they did not promote this fact to the emergent gravity picture.
Conclusion

- NC spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC.

- The emergent gravity from NC $U(1)$ gauge theory is the large $N$ duality and the emergent spacetime picture admits a background-independent formulation of quantum gravity.

- In order to understand NC spacetime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years.